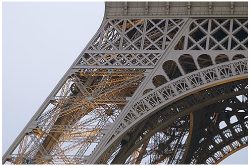
Numerical analysis of a topology optimization problem for the compliance of a linearly elastic structure

John Papadopoulos

17 April 2025

Brown University, METHODS Group Meeting

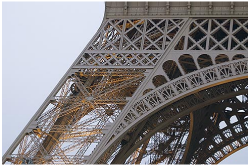
Topology optimization



(a) TO of compliance.

(b) TO of compliance.

Topology optimization



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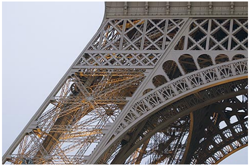

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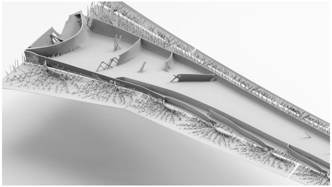
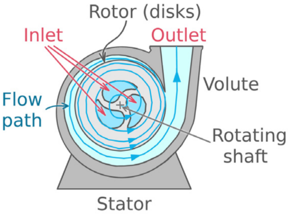
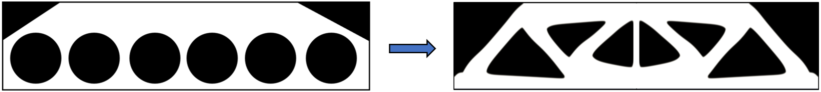


(d) Aage et al., *Nature* (2017).

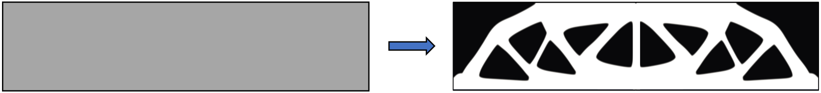


(e) Alonso et al., *CAMWA* (2019).

Shape vs. topology optimization



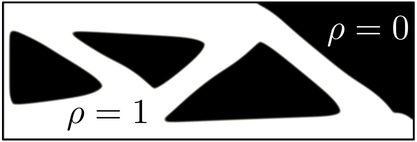
(a) Shape optimization



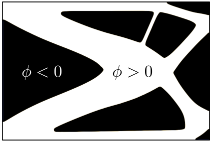
(b) Topology optimization

Models & optimization strategies

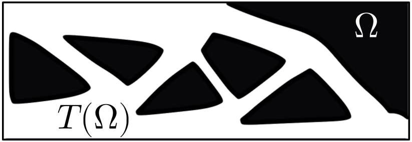
The model for representing the topology of the minimizer:



(a) Density.



(b) Level-set.



(c) Admissible domain maps.

The main textbook describing the density approach (Bendsoe, Sigmund, 2003) has $\sim 11,000$ citations. Over 20 professional software packages, consulting firms etc.

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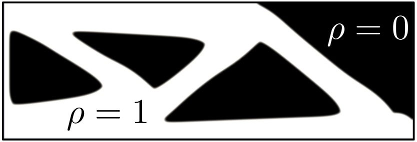

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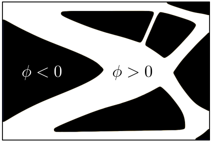

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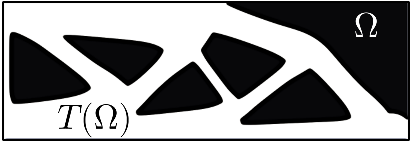

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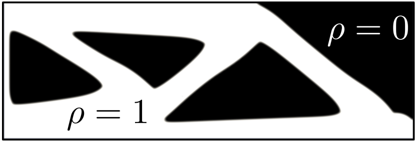

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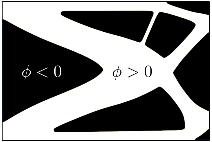

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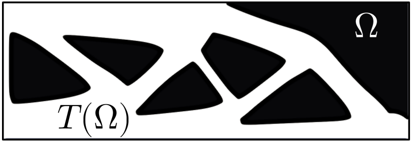

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Numerical difficulties

Models for topology optimization problems tend to:

- involve PDEs $\implies$ require a discretization, e.g. the finite element method (FEM).
- be nonconvex $\implies$ may support multiple local minima.

Open questions

- What is the best model?
- How do we interpret regions that are neither completely void or continuum?
- Do discretizations of the models actually converge to the minimizers of the original problem?
- Are the discretizations well behaved?
- Can we prove error bounds?
- Is there a general framework for proving convergence of FEM to all (density-based) topology optimization problems?

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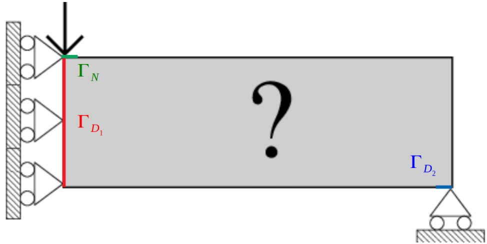
Compliance topology optimization



MBB beam.

A compliance problem

- Linear elasticity.
- Wish to minimize the compliance of the material (its displacement due to a force).
- Catch! We only have enough material to occupy 1/2 of the area.
- Requires solving a nonconvex optimzation problem with PDE, box, and volume constraints.

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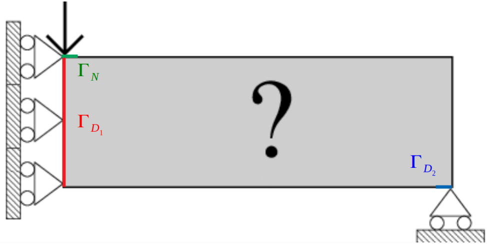

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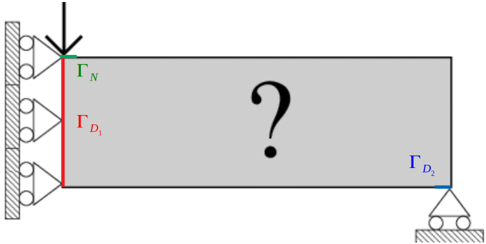

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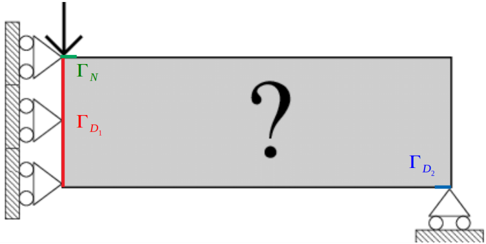

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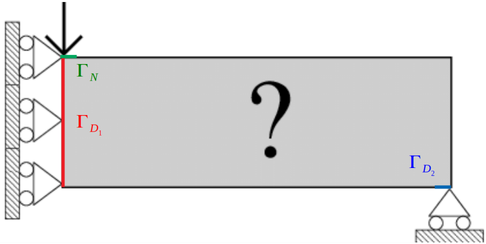

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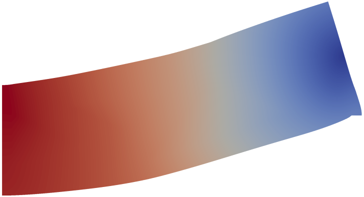
Topology optimization of elasticity

We are solving for the displacement $u \in H^1(\Omega; \mathbb{R}^d)$ and the density $\rho \in L^\infty(\Omega; [0,1])$.



Displacement: $u : \Omega \to \mathbb{R}^d$

Density: $\rho : \Omega \to [0,1]$

MBB Beam

MBB Optimization via LVPP

The SIMP model

Let $k(\rho) = \epsilon + (1-\epsilon)\rho^p$, $\epsilon \ll 1$, $p \geq 1$.

Optimization problem

$$\min_{u,\rho} \int_{\Gamma_N} f \cdot u \, \mathrm{d}s$$

subject to

$$-\mathrm{div}\sigma = 0,$$

$$\sigma = k(\rho)[2\mu\nabla_s(u) + \lambda\mathrm{div}(u)I] \qquad 0 \leq \rho \leq 1 \text{ a.e. in } \Omega,$$

$$u = 0 \text{ on } \Gamma_D \qquad \int_\Omega \rho \, \mathrm{d}x \leq \gamma|\Omega|.$$

$$\sigma\mathbf{n} = f \text{ on } \partial\Omega \backslash \Gamma_D.$$

μ and λ are the Lamé coefficients, $\nabla_s = (\nabla + \nabla^\top)/2$, I is the $d \times d$ identity matrix, and γ is the volume fraction.

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The SIMP function $k(\rho)$

$$k(\rho) = \epsilon + (1-\epsilon)\rho^p, \quad \epsilon \ll 1, \; p \geq 1.$$

Note that $k(1) = 1$ and $k(0) = \epsilon \ll 1$. So

$$\sigma \approx 2\mu\nabla_s(u) + \lambda\mathrm{div}(u)I \quad \text{wherever } \rho = 1 \text{ (high stiffness)},$$
$$\sigma \approx 0 \quad \text{wherever } \rho = 0 \text{ (no stiffness)}.$$

Role of the exponent p

Also as $p \to \infty$, this promotes $\rho(x) \to \{0, 1\}$, i.e. the density to become binary as intermediate regions (where $0 < \rho < 1$) become increasingly less optimal because $\rho^p \to 0$ as $p \to \infty$. A very common choice is $p = 3$.

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The SIMP model

Semi-bilinear form

$$a_\rho(u, v) = \int_\Omega k(\rho)[2\mu\nabla_s(u) : \nabla_s(v) + \lambda \mathrm{div}(u)\mathrm{div}(v)]\mathrm{d}x.$$

Variational formulation

Find $u \in H^1_{\Gamma_D}(\Omega)^d$, $\rho \in L^\infty(\Omega)$ that minimizes

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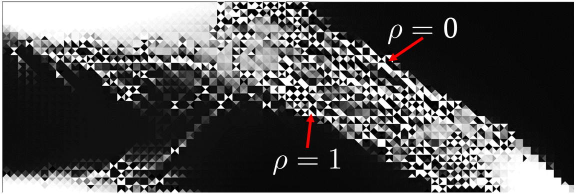
Existence of minimizers

Observation

When $p > 1$, the SIMP model does not guarantee the existence of a minimizer.

Consequence

After a FEM discretization, there exists a minimizer, but as $h \to 0$, we either get checkerboarding, or the beams of the elastic material become ever-thinner leading to nonphysical solutions in the limit.



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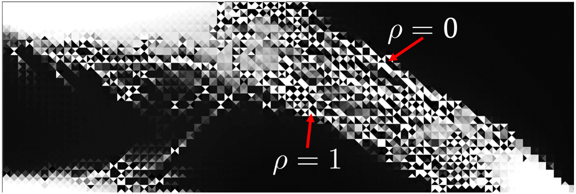

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Functional analysis

Strong convergence

$z_n \to z$ strongly in $L^q(\Omega)$ if $\lim_{n\to\infty} \|z_n - z\|_{L^q(\Omega)} = 0$.

Weak convergence

$z_n \rightharpoonup z$ weakly in $L^q(\Omega)$, if for all $v \in L^{q'}(\Omega)$, $1/q' + 1/q = 1$,

$$\int_\Omega z_n v \,\mathrm{d}x \to \int_\Omega z v \,\mathrm{d}x.$$

Weak-* convergence

$z_n \overset{*}{\rightharpoonup} z$ weakly-* in $L^\infty(\Omega)$, if for all $v \in L^1(\Omega)$, $\int_\Omega z_n v \,\mathrm{d}x \to \int_\Omega z v \,\mathrm{d}x$.

Weak convergence $\not\Rightarrow$ strong convergence

$\sin(nx) \rightharpoonup 0$ weakly in $L^2([0,2\pi])$, but $\|\sin(nx)\|_{L^2([0,2\pi])} = \pi \;\forall\; n \in \mathbb{Z}_+$.

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$$\int_\Omega z_n v \,\mathrm{d}x \to \int_\Omega z v \,\mathrm{d}x.$$

Weak-* convergence

$z_n \overset{*}{\rightharpoonup} z$ weakly-* in $L^\infty(\Omega)$, if for all $v \in L^1(\Omega)$, $\int_\Omega z_n v \,\mathrm{d}x \to \int_\Omega z v \,\mathrm{d}x$.

Weak convergence $\not\Rightarrow$ strong convergence

$\sin(nx) \rightharpoonup 0$ weakly in $L^2([0, 2\pi])$, but $\|\sin(nx)\|_{L^2([0,2\pi])} = \pi \;\forall\; n \in \mathbb{Z}_+$.

What goes wrong?

Minimizing sequence

Extract a minimizing sequence (u_n, ρ_n) such that

$$u_n \rightharpoonup \hat{u} \text{ weakly in } H^1(\Omega)^d$$

$$\rho_n \overset{*}{\rightharpoonup} \hat{\rho} \text{ weakly-* in } L^\infty(\Omega)$$

Problem

However the weak-* convergence means that

$$\lim_{n\to\infty} a_{\rho_n}(u_n, v) \neq a_\rho(u, v) = (f, v)_{L^2(\Gamma_N)}.$$

One cannot take the limit in the PDE constraint!

Solution

Somehow extract a stronger converging sequence for ρ_n.

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Sobolev regularization

Modify objective functional. For some $\delta \ll 1$ and $q \in [1, \infty]$, find (u_δ, ρ_δ) minimizing

$$\min_{u,\rho} \int_{\Gamma_N} f \cdot u \, \mathrm{d}s + \frac{\delta}{q} \|\nabla \rho\|^q_{L^q(\Omega)} + \text{rest of constraints.}$$

Then we extract a minimizing sequence $\rho_n \rightharpoonup \hat{\rho}$ weakly in $W^{1,q}(\Omega) \implies a_{\rho_n}(u_n, v) \to a_\rho(u, v) = (f, v)_{L^2(\Gamma_N)}$.

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Density filtering

Modify PDE constraint. Consider $F \in W^{1,\infty}(\mathbb{R}^d)$, $F \geq 0$, $\|F\|_{L^1(\mathbb{R}^d)} = 1$. E.g.

$$F(x) = \frac{\exp(\|x\|^2/(2\sigma^2))}{\|\exp(\|\cdot\|^2/(2\sigma^2))\|_{L^1(\mathbb{R}^d)}}.$$

We define the *filtered* density $\tilde{\rho}(\rho) \in W^{1,\infty}(\Omega)$ as

$$\tilde{\rho}(\rho)(x) = (F \star \rho)(x) = \int_\Omega F(x-y)\rho(y)\,\mathrm{d}y,$$

and instead solve

$$a_{\tilde{\rho}(\rho)}(u,v) = (f,v)_{L^2(\Gamma_N)}.$$

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Restriction methods

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Finite element discretization

Quasi-uniform and non-degenerate triangulation.

$$\mathcal{H} := \{\eta \in L^\infty(\Omega) : 0 \le \eta \le 1, \|\eta\|_{L^1(\Omega)} \le \gamma|\Omega|\}.$$

Conforming discretization

$$u_h \in X_h \subset H^1(\Omega)^d,$$

$$\rho_h \in \mathcal{H}_h \subset \begin{cases} \mathcal{H} & \text{density filtering,} \\ W^{1,q}(\Omega) \cap \mathcal{H} & \text{Sobolev regularization.} \end{cases}$$

Discretized filtered density:

$$\tilde{\rho}_h(\rho_h)(x) = \Pi_h \int_\Omega F(x-y)\rho_h(y)\,\mathrm{d}y.$$

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(Brief) history of FEM convergence

Density filtering

There exists a minimizer (u, ρ) and a sequence such that

$$u_h \to u \text{ strongly in } H^1(\Omega)^d,$$
$$\rho_h \overset{*}{\rightharpoonup} \rho \text{ weakly-* in } L^\infty(\Omega),$$
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Open problems

1. What is (u, ρ)? Is it a local or global minimum? What about the other minima?
2. Does $\rho_h \to \rho$ strongly?
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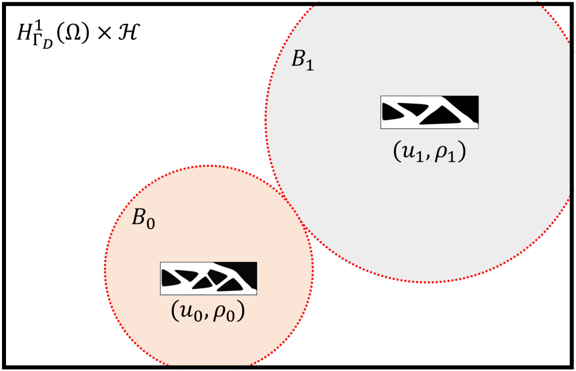
Finite element convergence

Key idea: fix an isolated local minimizer (u, ρ).

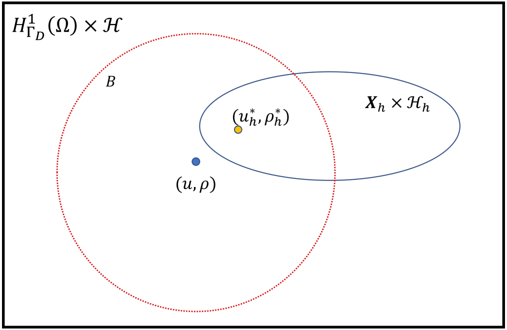


Finite element convergence

Consider the modified finite-dimensional optimization problem:

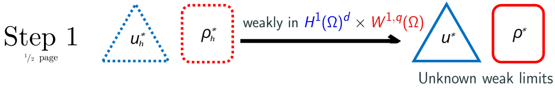
$$\text{Find a compliance minimizer } (u_h^*, \rho_h^*) \in B \cap (X_h \times \mathcal{H}_h). \qquad (*)$$

(u_h^*, ρ_h^*) is not computable in practice.



Finite element convergence: Sobolev regularization

Find a discretized compliance minimizer $(u_h^*, \rho_h^*) \in B \cap (X_h \times \mathcal{H}_h)$. (*)



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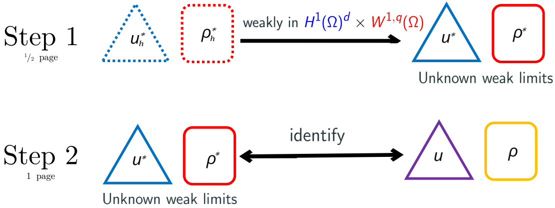

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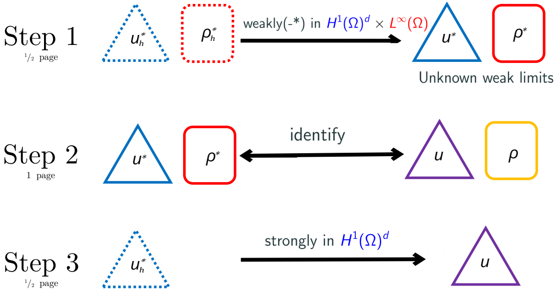

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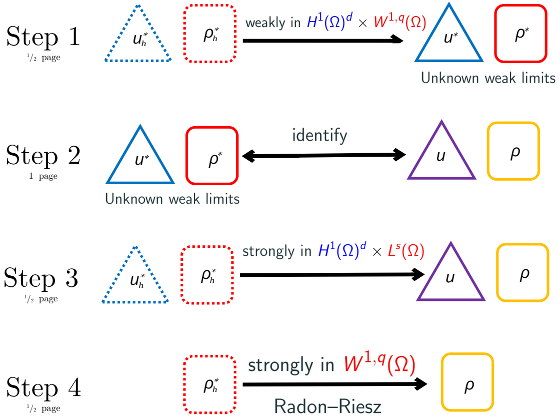


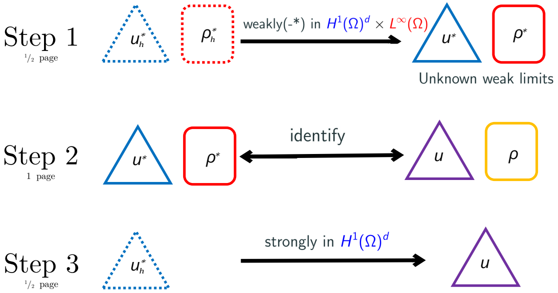
Finite element convergence: Sobolev regularization

Find a discretized compliance minimizer $(u_h^*, \rho_h^*) \in B \cap (X_h \times \mathcal{H}_h)$. (*)

Strong convergence of u_h^* and ρ_h^*, lifts the basin of attraction constraint, i.e. no more dependence on B.

Finite element convergence: Density filtering

Find a discretized compliance minimizer $(u_h^*, \rho_h^*) \in B \cap (X_h \times \mathcal{H}_h)$. (*)



Strong convergence of ρ_h^* in $L^s(\Omega)$, $s \in [1, \infty)$ and $\tilde{\rho}_h$ in $W^{1,q}(\Omega)$ is subtle.

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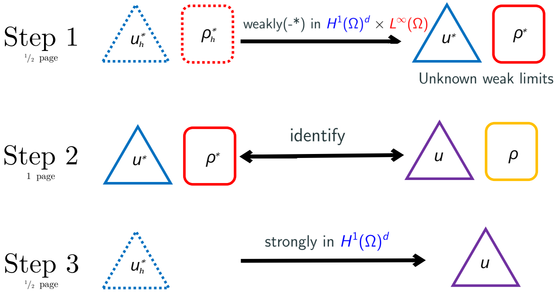

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Density filtering: strong convergence of ρ_h^*

ϵ-perturbed problem: find $(u_\epsilon, \rho_\epsilon) \in B \cap (H^1(\Omega)^d \times \mathcal{H})$.

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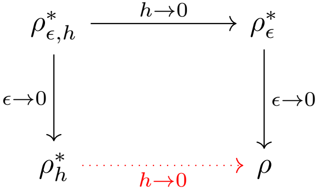

Figure 5: $\to$: strong convergence in $L^2(\Omega)$.

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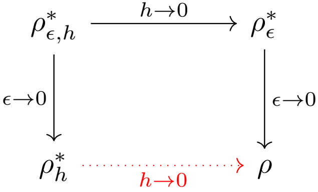

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Outline of proof

1. **Estimates** $\Rightarrow \rho^*_{\epsilon,h} \to \rho^*_\epsilon$ strongly in $L^2(\Omega)$ as $h \to 0$.
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2. **Minimizer** $\Rightarrow \rho^*_\epsilon \to \rho$, $\rho^*_{\epsilon,h} \to \rho^*_h$ strongly in $L^2(\Omega)$ as $\epsilon \to 0$.
3. **Boundedness** $\Rightarrow$ $\lim_{h\to 0} \lim_{\epsilon \to 0} \|\rho^*_{\epsilon,h}\|_{L^2(\Omega)} = \lim_{\epsilon\to 0} \lim_{h \to 0} \|\rho^*_{\epsilon,h}\|_{L^2(\Omega)}$.
4. **Interchange of limits** $\Rightarrow \lim_{h\to 0} \|\rho^*_h\|_{L^2(\Omega)} = \|\rho\|_{L^2(\Omega)}$.
5. **Radon–Riesz**. (4) $+ \rho^*_h \rightharpoonup \rho$ in $L^2(\Omega) \implies \rho_h \to \rho$ strongly in $L^2(\Omega)$.
6. **Consequence**. Strong convergence of u^*_h and ρ^*_h, lifts the basin of attraction constraint, i.e. no more dependence on B.

Density filtering: strong convergence of ρ_h^*

ϵ-perturbed problem: find $(u_\epsilon, \rho_\epsilon) \in B \cap (H^1(\Omega)^d \times \mathcal{H})$

$$\min_{u,\rho} (f, u)_{L^2(\Gamma_N)} + \frac{\epsilon}{2}\|\rho\|^2_{L^2(\Omega)} + \text{PDE constraint.}$$

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1. **Estimates** $\Rightarrow \rho^*_{\epsilon,h} \to \rho^*_\epsilon$ strongly in $L^2(\Omega)$ as $h \to 0$.
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Density filtering: strong convergence of $\tilde{\rho}_h$

ϵ-perturbed problem: find $(u_\epsilon, \rho_\epsilon) \in B \cap (H^1(\Omega)^d \times \mathcal{H})$

$$\min_{u,\rho} (f, u)_{L^2(\Gamma_N)} + \frac{\epsilon}{p} \|\nabla \tilde{\rho}(\rho)\|^p_{L^p(\Omega)} + \text{PDE constraint.}$$

$$\begin{array}{ccc}
\nabla \tilde{\rho}_h(\rho^*_{\epsilon,h}) & \xrightarrow{h \to 0} & \nabla \tilde{\rho}(\rho^*_\epsilon) \\
\Big\downarrow {\scriptstyle \epsilon \to 0} & & \Big\downarrow {\scriptstyle \epsilon \to 0} \\
\nabla \tilde{\rho}_h(\rho^*_h) & \overset{}{\underset{h \to 0}{\dashrightarrow}} & \nabla \tilde{\rho}(\rho)
\end{array}$$

One deduces that $\tilde{\rho}_h(\rho_h) \to \tilde{\rho}(\rho)$ strongly in $W^{1,q}(\Omega)$.

Density filtering: strong convergence of $\tilde{\rho}_h$

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Conclusions

- All isolated minimizers are approximated by FEM.
- Displacements converge strongly $u_h \to u$ in $H^1(\Omega)^d$.
- Density filtering: density converges strongly $\rho_h \to \rho$ in $L^s(\Omega)$, $s \in [1, \infty)$.
- Density filtering: filtered density converges strongly $\tilde{\rho}_h(\rho_h) \to \tilde{\rho}(\rho)$ in $W^{1,q}(\Omega)$.
- Sobolev regularization: density converges strongly $\rho_h \to \rho$ in $W^{1,q}(\Omega)$.

For more details see:

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Thank you for listening!

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