Imperial College London

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Computing multiple solutions of topology optimization problems

USNCCM17




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[1]Imperial College London; [2]University of Oxford; [3]Simula Research Institute

Topology optimization

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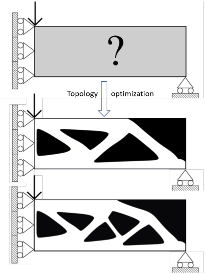
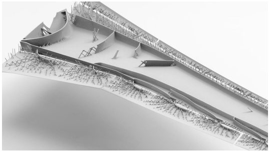
Objective

Find the optimal distribution of a continuum that minimizes a problem-specific cost functional with no prior knowledge of the optimal shape or topology.



Two minimizers of the same optimization problem



Aage, Andreassen, Lazarov, Sigmund, *Nature* (2017)

Models for topology optimization problems tend to:

- involve PDEs $\implies$ require a discretization;
- be nonconvex $\implies$ may support multiple local minima.

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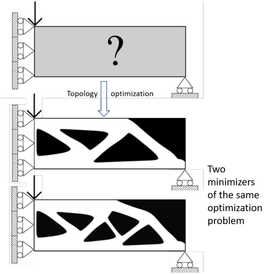

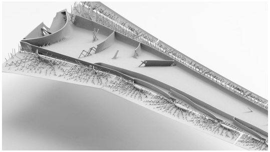

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Choice of optimization strategy

Observations

- Potentially many (local) minimizers.
- Millions of degrees of freedom.

Consequences

- Require quickly converging algorithms.
- Compute multiple minimizers in a systematic manner.
- Require preconditioners for the solves e.g. effective multigrid cycles.

Our proposal

The deflated barrier method.

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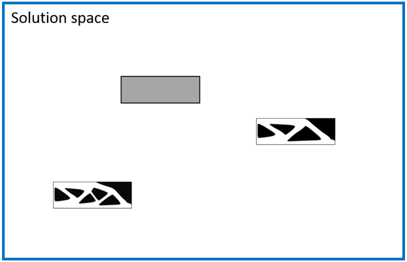
Deflated barrier method

Barrier-like terms + primal-dual active set strategy + deflation



The deflated barrier method

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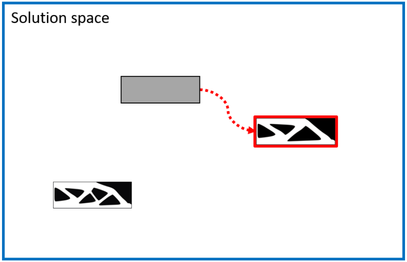
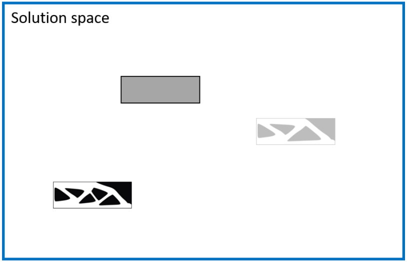


Step I: optimize from initial guess

The deflated barrier method

Deflated barrier method

Barrier-like terms + primal-dual active set strategy + deflation



Step II: deflate solution found

The deflated barrier method

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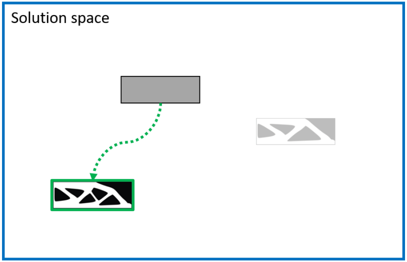

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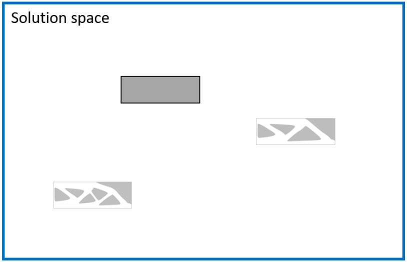

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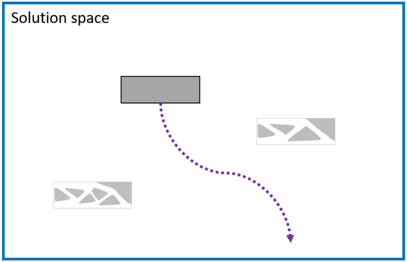

Step III: termination on nonconvergence

Construction of deflated problems

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A nonlinear transformation of first-order optimality conditions

$$\mathcal{F}(z) = 0 \to \mathcal{G}(z) \coloneqq \mathcal{M}(z;r)\mathcal{F}(z) = 0.$$

A deflation operator

We say that $\mathcal{M}(z;r)$ is a deflation operator if for any sequence $z \to r$

$$\liminf_{z \to r} \|\mathcal{G}(z)\| = \liminf_{z \to r} \|\mathcal{M}(z;r)\mathcal{F}(z)\| > 0.$$

Theorem

This is a deflation operator for $p \geq 1$:

$$\mathcal{M}(z;r) = \left(\frac{1}{\|z-r\|^p} + 1\right).$$

Construction of deflated problems

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Deflation is *very* easy!

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Step 1

Compute the normal *undeflated* Newton update δz.

Step 2

Let $m = m(z_k) = \mathcal{M}(z_k, r)$. Then the deflated Newton update is

$$\delta z_D = \tau(z_k, \delta z)\delta z$$

where

$$\tau(z_k, \delta z) := \left(1 + \frac{m^{-1}(m')(\delta z)}{1 - m^{-1}(m')(\delta z)}\right).$$

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The Borrvall–Petersson problem

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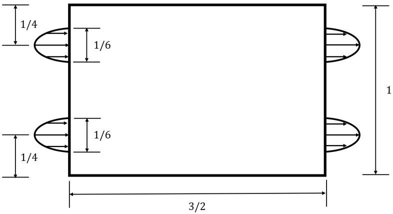
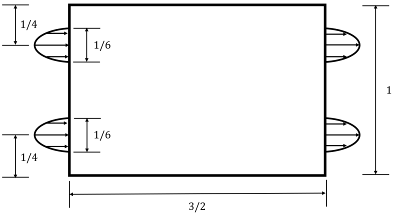


Double-pipe problem

A fluid topology optimization problem

- Stokes flow.
- Wish to minimize the power dissipation of the flow;
- Catch! The channels can occupy up to 1/3 area.
- Requires solving a nonconvex optimization problem with PDE, box, and volume constraints.

The Borrvall–Petersson problem



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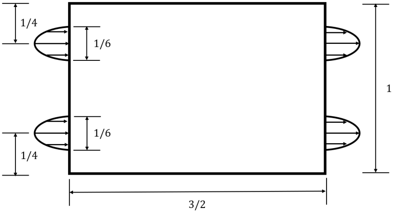

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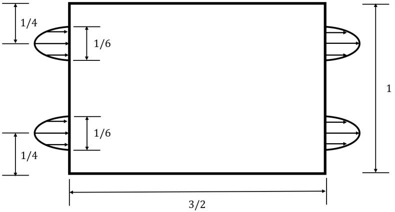

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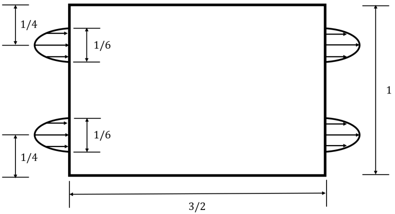

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Double-pipe solutions

(a) Straight channels

Double-pipe solutions

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(a) Straight channels

(b) Double-ended wrench

Double-pipe solutions

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(a) Straight channels

(b) Double-ended wrench

(c) Neumann (i)

Double-pipe solutions

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(a) Straight channels

(b) Double-ended wrench

(c) Neumann (i)

(d) Neumann (ii)

Borrvall & Petersson

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Problem

Find velocity u and material distribution ρ that minimize

$$J(u, \rho) = \frac{1}{2} \int_\Omega \alpha(\rho)|u|^2 + |\nabla u|^2 - 2f \cdot u \, \mathrm{d}x,$$

subject to $\mathrm{div}(u) = 0$, $0 \leq \rho \leq 1$, and $\int_\Omega \rho \, \mathrm{d}x \leq \gamma|\Omega|$.

Deflated barrier method

For $\mu = \mu_0$ ($\mu \to 0$), solve $\nabla L_\mu(u, \rho, p, \lambda)$ "=" 0 with a primal-dual active set strategy where

$$L_\mu(u, \rho, p, \lambda) = J(u, \rho) - \int_\Omega p \mathrm{div}(u) + \lambda(\gamma - \rho) \, \mathrm{d}x - \mu \int_\Omega \log((\rho + \epsilon)(1 + \epsilon - \rho)) \, \mathrm{d}x.$$

Borrvall & Petersson

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The deflated barrier method

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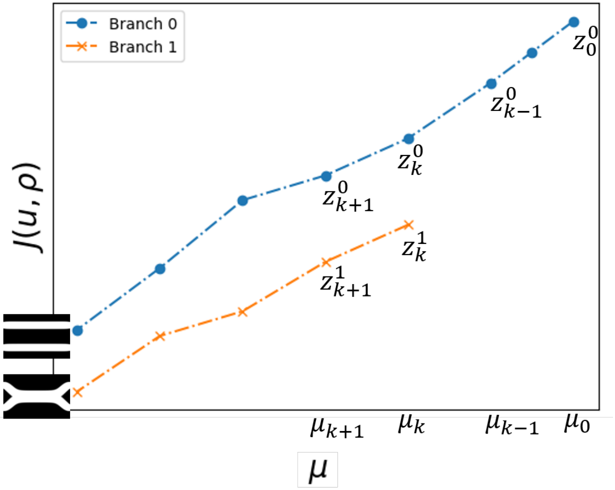

A fluid topology optimization problem

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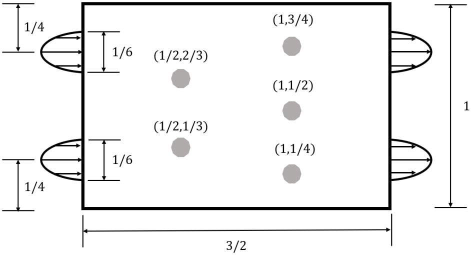

Five-holes double-pipe setup.

Fluid topology optimization

- Navier–Stokes flow.
- Wish to minimize the power dissipation of the flow;
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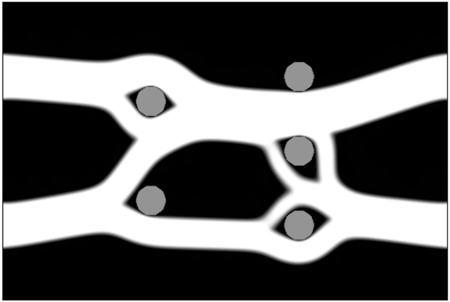
A fluid topology optimization problem



$$J = 60.09$$

A fluid topology optimization problem

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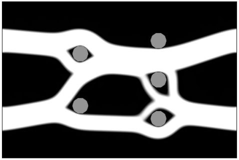

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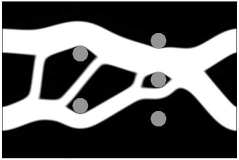
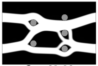


$J = 57.96$

A fluid topology optimization problem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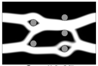
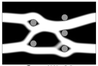
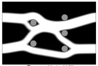
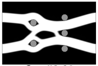
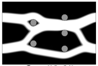

$J = 56.27$


$J = 55.60$


$J = 54.57$


$J = 53.20$


$J = 53.25$

A fluid topology optimization problem

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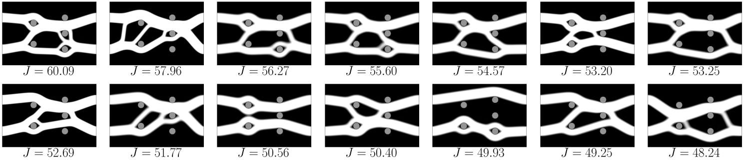

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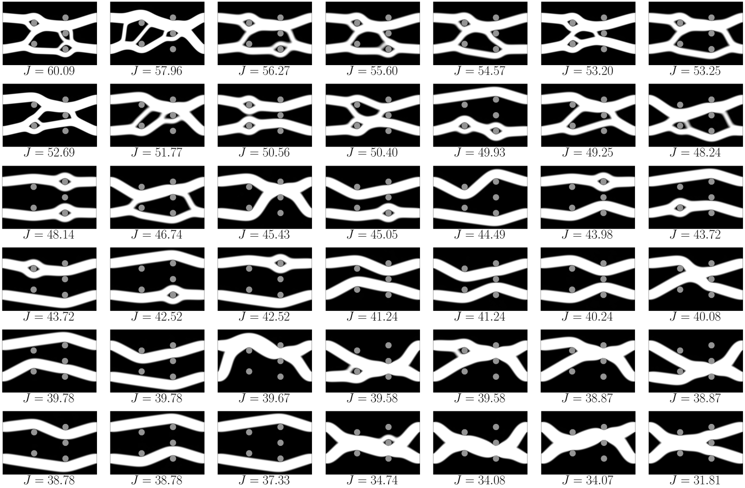
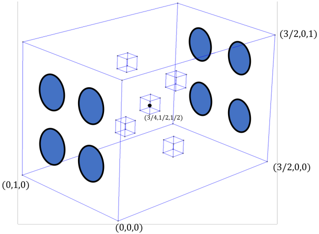


Examples

- 3D discretization on a $40 \times 40 \times 40$ block $\sim$ 3,000,000 dofs.
- (Stokes) Nevertheless still numerically tractable via preconditioning techniques implemented with `Firedrake`
  - Nested block preconditioning via Schur complements;
  - Augmented Lagrangian control of the pressure Schur complement;
  - Vertex-star patch type relaxation for multigrid schemes.



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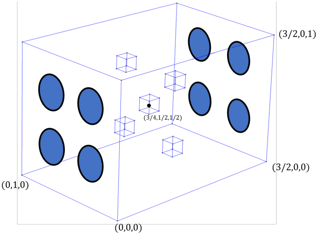

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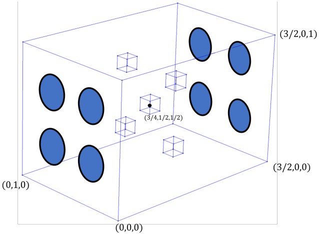

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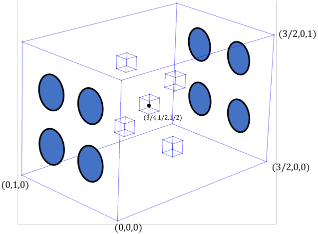

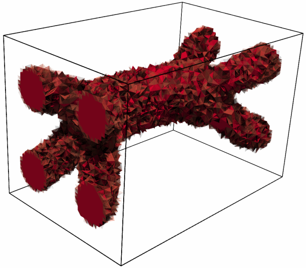

3D five-holes quadruple-pipe

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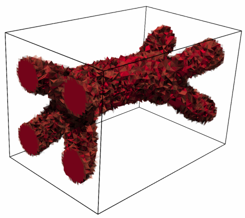

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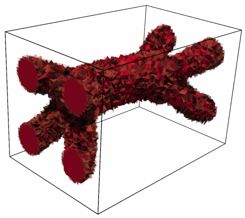

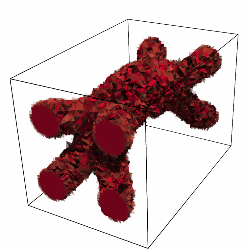

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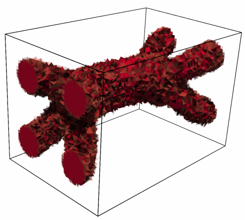

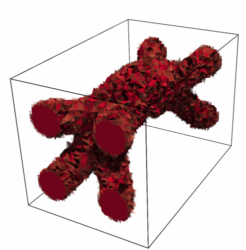

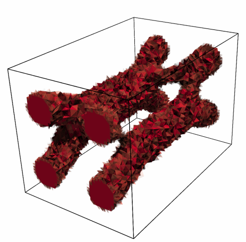

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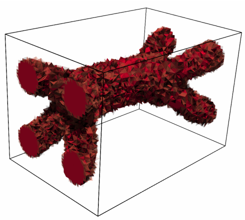
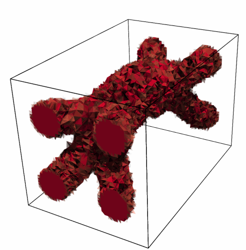
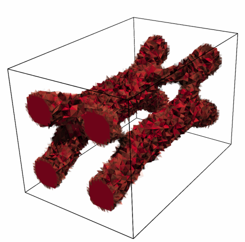
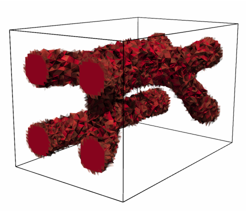
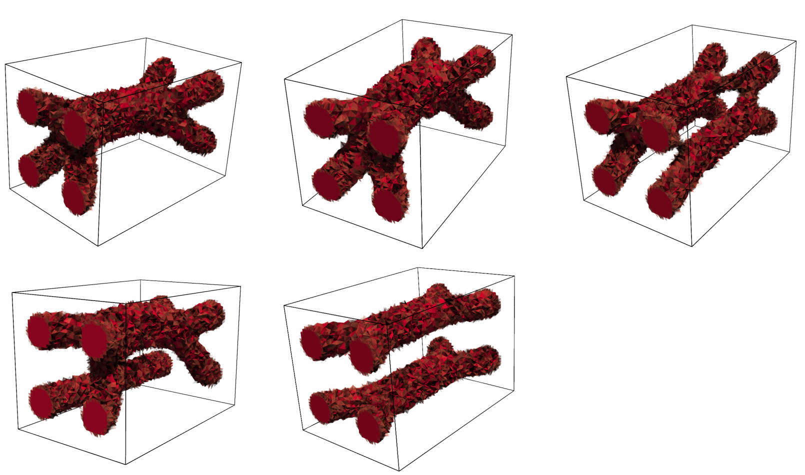




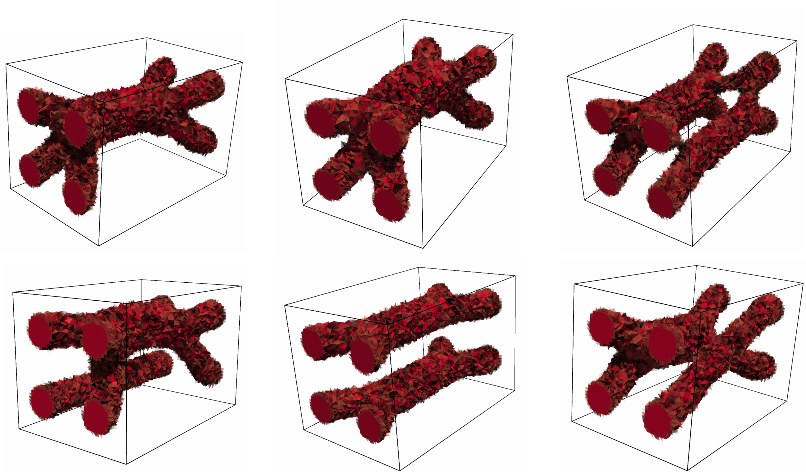




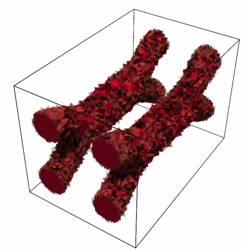
3D five-holes quadruple-pipe



3D five-holes quadruple-pipe



3D five-holes quadruple-pipe



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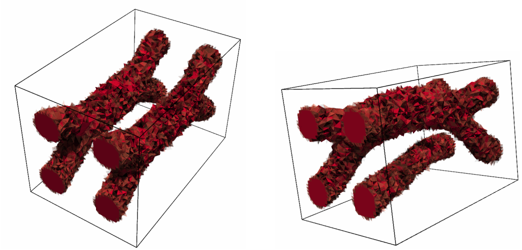

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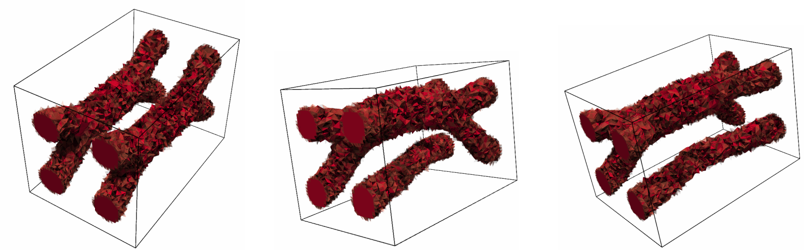

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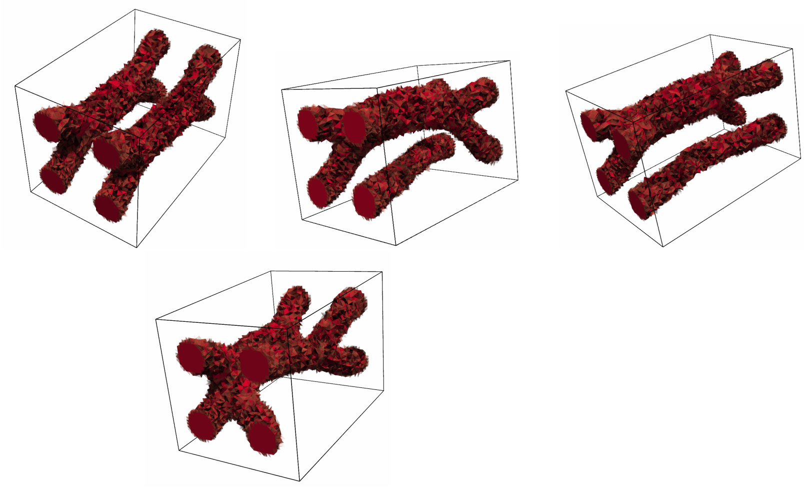

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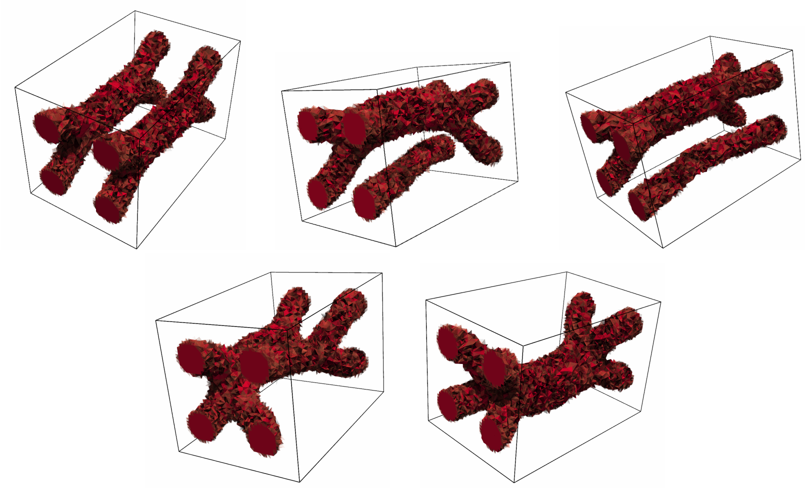

Further refinement

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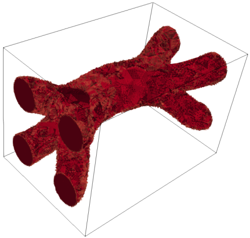

15,953,537 degrees of freedom.

Further refinement

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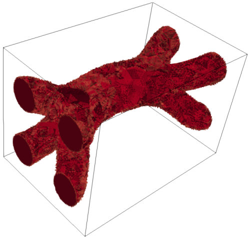

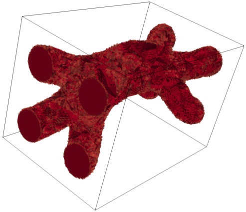

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Further refinement

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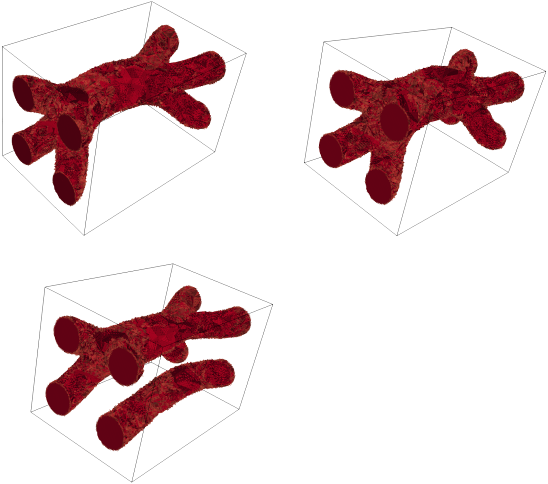

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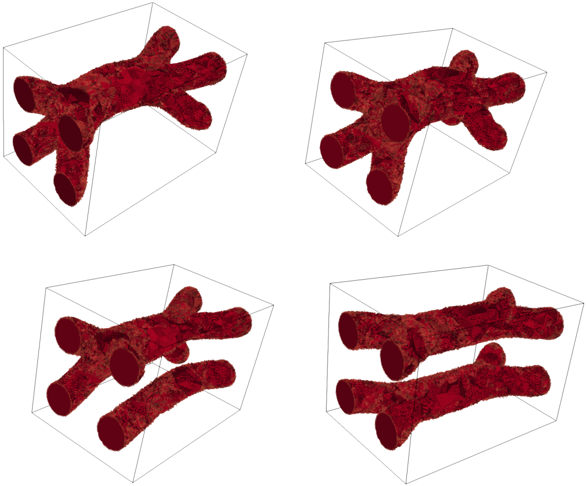
Further refinement



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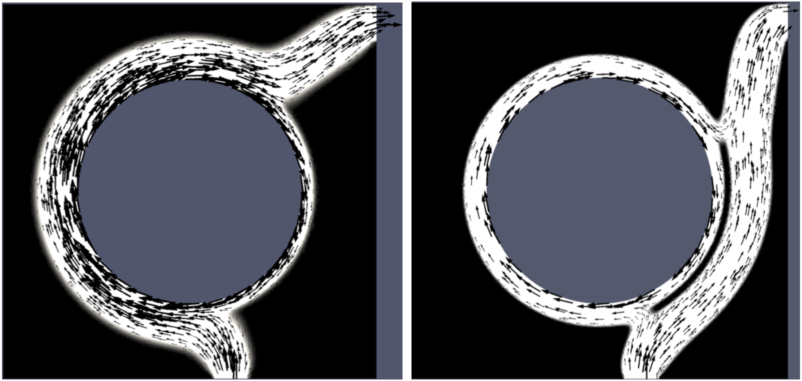
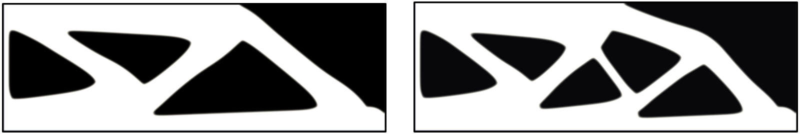
More examples



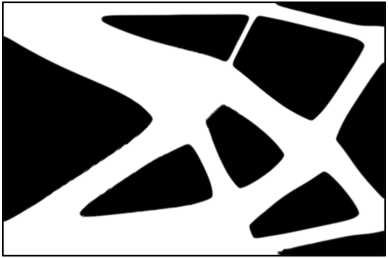
Roller pump

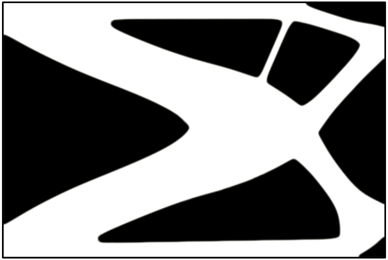
More examples



MBB beam

More examples





Double cantilever

Conclusions

- A strategy for computing multiple solutions of topology optimization problems.
- Barrier-like terms + active set strategy + deflation.
- Can solve large 3D problems with good preconditioners.

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Software

Deflated barrier method

https://github.com/ioannisPApapadopoulos/fir3dab.

Deflation

https://github.com/ioannisPApapadopoulos/Deflation.

Deflation for bifurcation diagrams

https://bitbucket.org/pefarrell/defcon.

Thank you for listening!

✉ ioannis.papadopoulos13@imperial.ac.uk