A sparse hierarchical *hp*-finite element method on disks, annuli, and cylinders

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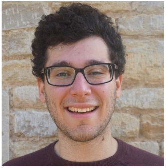


Helmholtz/screened Poisson equation

Problem statement

Let Ω be a disk, annulus or cylinder and $\lambda : \Omega \to \mathbb{R}$. We want to find u satisfying

$$(-\Delta + \lambda)u = f, \quad u|_{\partial\Omega} = 0.$$

A solver that delivers:

- Fast convergence even with radial discontinuities in f or λ.
- Symmetric and sparse linear systems.
- A "fast" $O(p^d \log p)$ quasi-optimal complexity solve.

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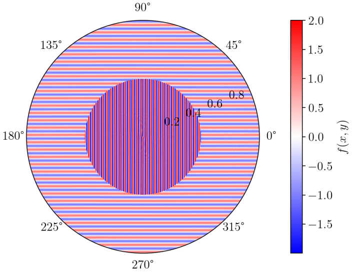
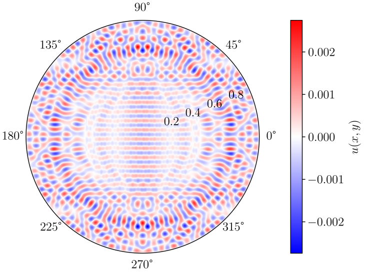
Example: high-frequency Helmholtz

Let $\Omega =$ unit disk. Solve

$$(-\Delta - 80^2)u(x,y) = f(x,y) := \begin{cases} 2\sin(200x) & r \leq 1/2, \\ \sin(100y) & r > 1/2, \end{cases} + \text{ zero bcs.}$$



(a) $f(x,y)$.



(b) $u(x,y)$.

Note the lack of Runge phenomenon or numerical artefacts at $r = 0$ and $r = 1/2$.

Discretization

Discretization

Approximate the solution u of $(-\Delta + \lambda)u = f$ by solving a finite-dimensional linear system $(A + M_\lambda)\mathbf{u} = \mathbf{b}$.

Finite element method (FEM)

Pick a finite-dimensional basis $\{\phi_j\}$.

$$A_{ij} = (\nabla\phi_j, \nabla\phi_i)_{L^2(\Omega)}, \quad [M_\lambda]_{ij} = (\phi_j, \lambda\phi_i)_{L^2(\Omega)}, \quad \mathbf{b}_i = (f, \phi_i)_{L^2(\Omega)}.$$

A and M_λ are symmetric.

Goal

Pick a FEM basis such that:

- A and M_λ are sparse even for high p
- fast quasi-optimal quadrature
- fast convergence
- fast quasi-optimal solves

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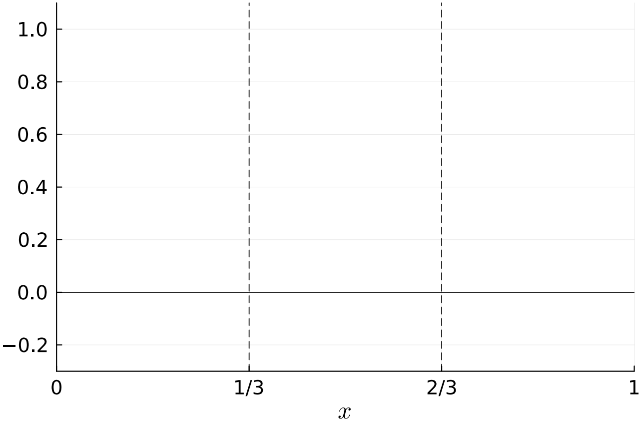
A hierarchical FEM basis in 1D



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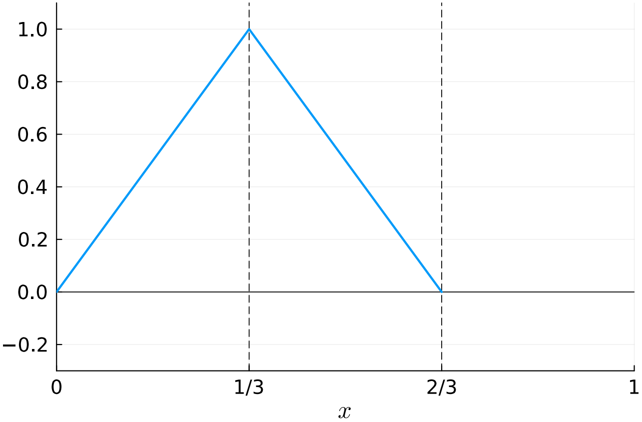

Begin with adding classical FEM piecewise linear “hat” polynomials.

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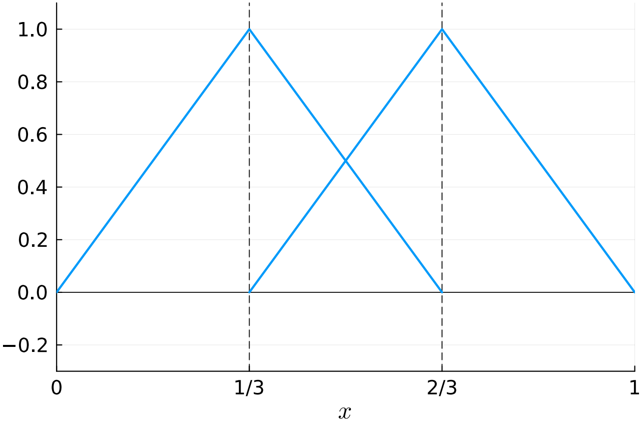

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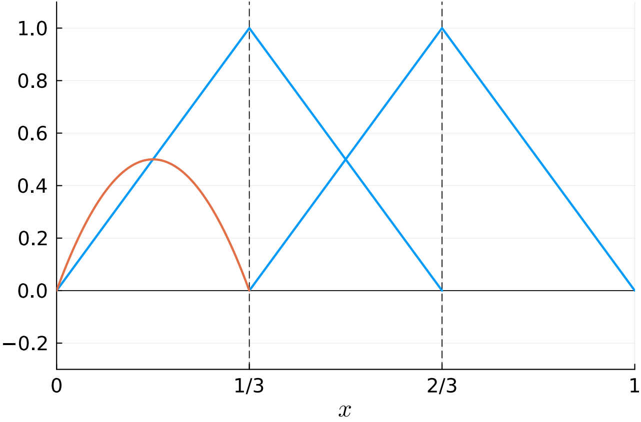

Next add quadratic “bubble” polynomials only supported on one element each.

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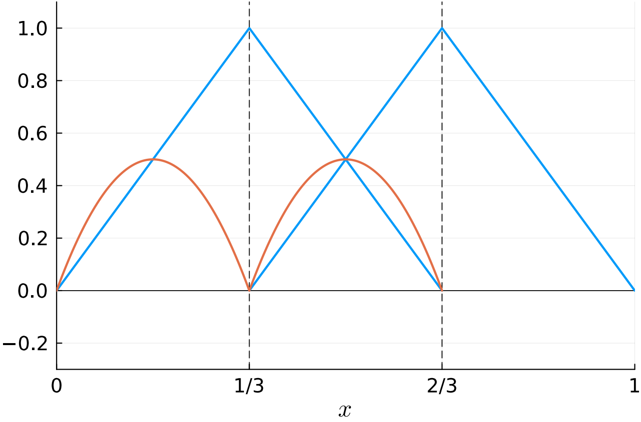

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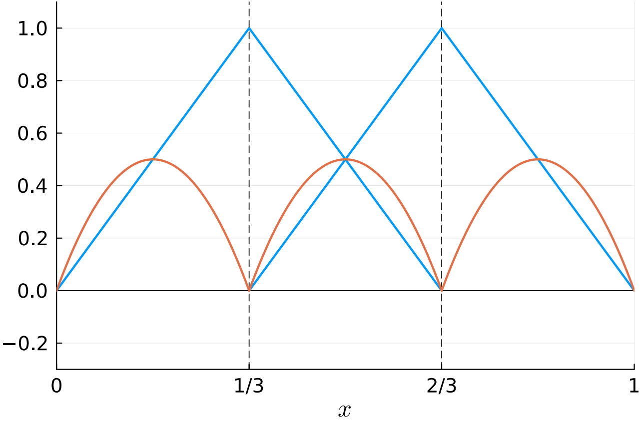

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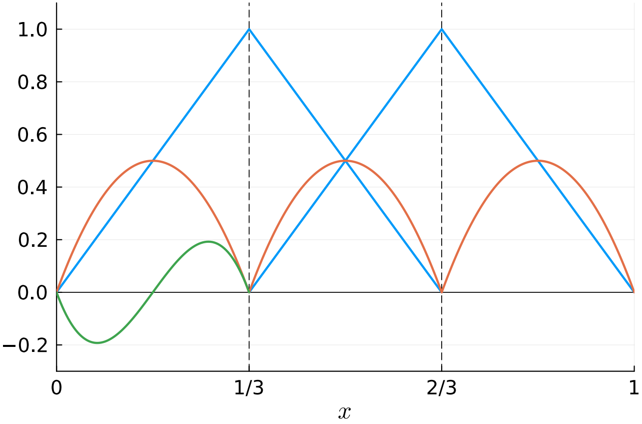

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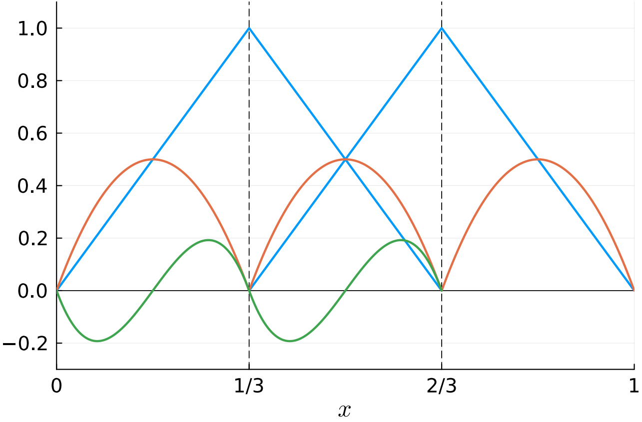

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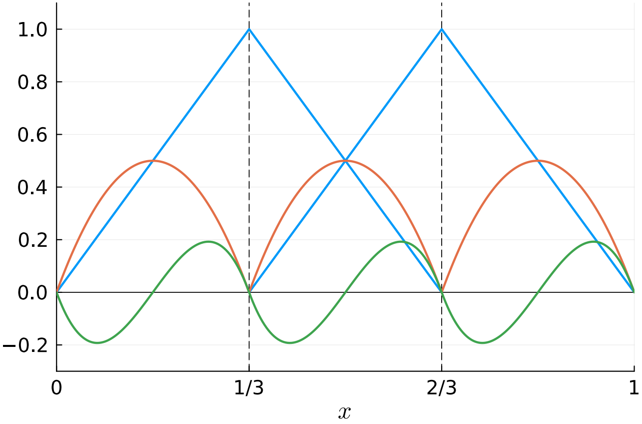

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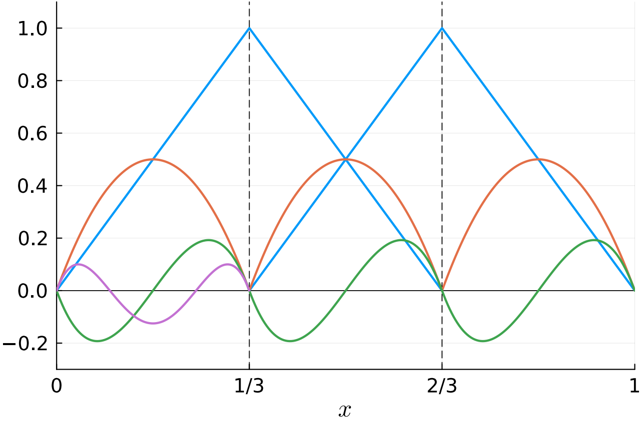

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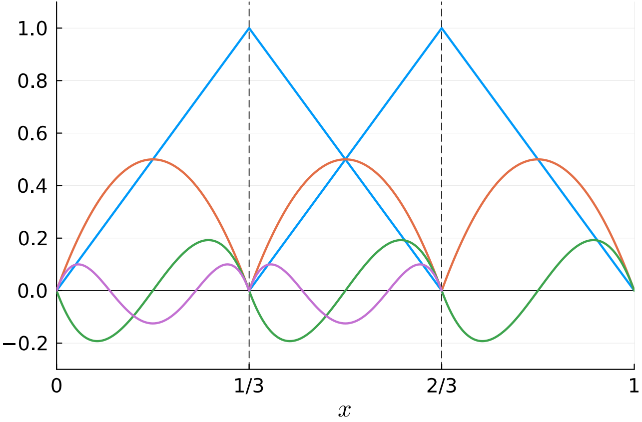

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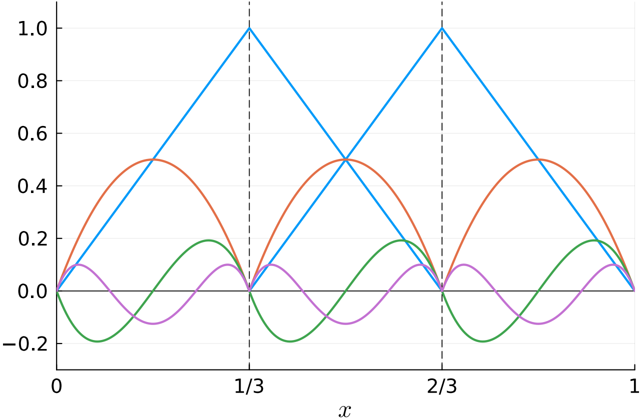

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Orthogonal polynomials for the disk & annulus

Zernike polynomials

Zernike polynomials are multivariate polynomials in x and y orthogonal on the unit disk with respect to $(1-r^2)^a$ for a user-chosen $a \geq 0$.

Zernike annular polynomials

Zernike annular polynomials are multivariate polynomials in x and y orthogonal on the annulus $\{0 < \rho \leq r \leq 1\}$ with respect to $(1-r^2)^a(r^2-\rho^2)^b$ for a user-chosen inner-radius $\rho > 0$ and $a, b \geq 0$.

Fast transforms (2024)

Zernike (annular) polynomials now have fast quasi-optimal complexity expansion and evaluation transforms via Cholesky and QR factorizations of tridiagonal Jacobi matrices.

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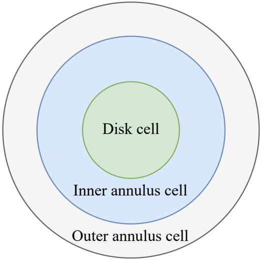

This mesh combined with Zenrike (annular) polynomials preserves the Fourier mode decoupling and captures any potential radial discontinuities.

Hierarchical FEM basis for disks and annuli

Zernike (annular) polynomials allow one to extend the 1D hats and bubbles principle to disks and annuli. For each Fourier mode, we can define hat polynomials supported on a maximum of two cells and high-order bubble polynomials supported on a maximum of one cell.

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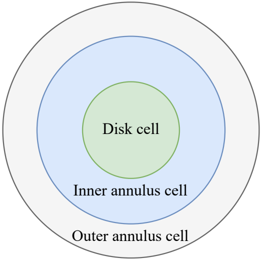

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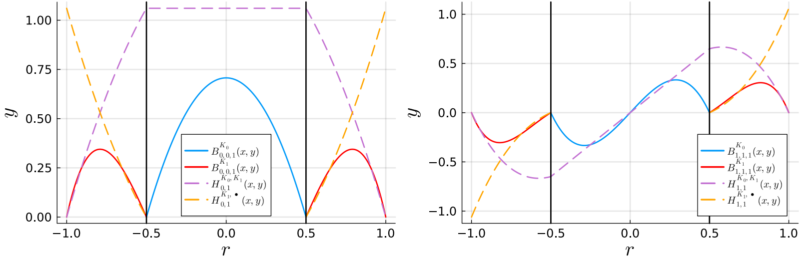
Hats and bubbles for the disk



(a) Fourier mode $(0, 1)$ (b) Fourier mode $(1, 1)$

Slice at $\theta = 0$ of the hat and bubble functions on the unit disk meshed into $\{0 \leq r \leq 1/2\}$ and $\{1/2 \leq r \leq 1\}$.

Sparsity

This hierarchical Zernike FEM basis leads to block-diagonal stiffness/mass matrices:

$$A = \begin{pmatrix} A_0 & & \\ & A_1 & \\ & & \ddots \end{pmatrix}, \quad M = \begin{pmatrix} M_0 & & \\ & M_1 & \\ & & \ddots \end{pmatrix}.$$

The individual blocks have a sparse arrowhead matrix structure:

Leibniz Association

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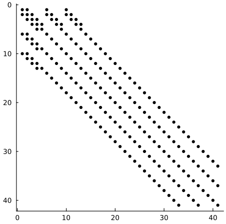

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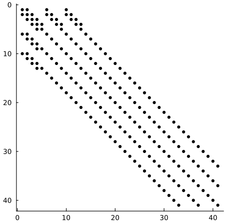

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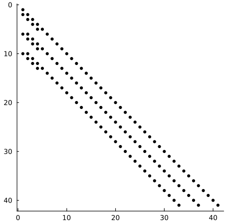


(b) Lower reverse Cholesky factor

Complexity flowchart

Expansion coefficients of $f(x, y)$: $O(np^2 \log p)$

Compute entries of A and M_λ: $O(np^2)$

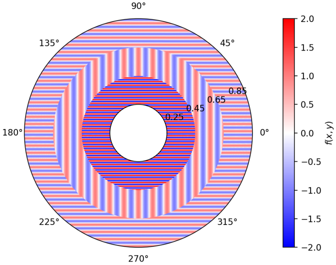
Solve $(A + M_\lambda)\mathbf{u} = \mathbf{b}$: $O(np^2)$

Evaluate $u_{hp}(x, y)$ on a grid: $O(np^2 \log p)$

Overall quasi-optimal complexity: $O(np^2 \log p)$.

Example: high-frequency on an annulus

Let $\Omega = \{1/4 \leq r \leq 1\}$. Find u satisfying $-\Delta u - 80^2 u = f$.



$f(x, y)$.

$$f(x,y) = \begin{cases} 2\sin(200y) & 1/4 < r \leq 1/2, \\ \cos(50x) & 1/2 < r \leq 3/4, \\ \sin(100y) & 3/4 < r \leq 1. \end{cases}$$

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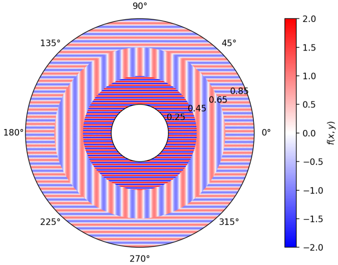

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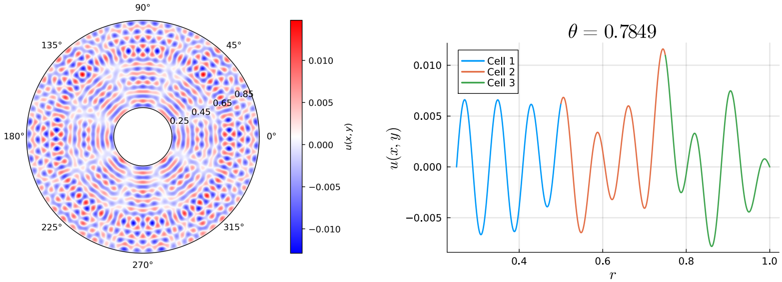
$f(x, y)$.

3 cells: $\{1/4 \leq r \leq 1/2\}$, $\{1/2 \leq r \leq 3/4\}$, and $\{3/4 \leq r \leq 1\}$.

Example: high-frequency on an annulus



Truncation degree, $p = 200$.

Example: quantum harmonic oscillator

Let $\Omega = \{0 \leq r \leq 50\}$. Find u satisfying

$$\mathrm{i}\partial_t u = (-\Delta + r^2)u, \quad u(x, y, 0) = \psi_{20,21}(x, y).$$

We discretize in time with Crank–Nicolson and then mesh the domain into 16 cells with $p = 100$ on each cell reducing the solve to

$$(2M + \mathrm{i}\delta t(A + M_{r^2}))\mathbf{u}_{k+1} = (2M - \mathrm{i}\delta t(A + M_{r^2}))\mathbf{u}_k.$$

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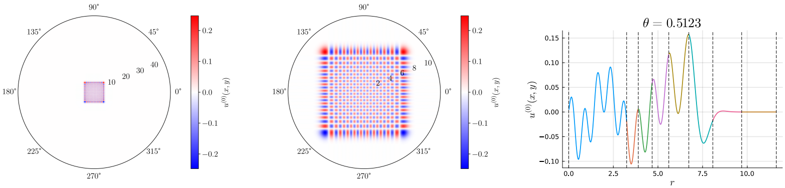
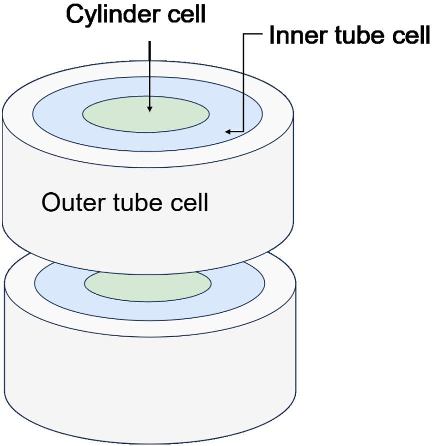


Solution after a full period. Error $\approx 10^{-6}$.

Example: 3D cylinder

Basis: *hp*-FEM for disk $\otimes$ *p*-FEM 1D basis.



Example: 3D cylinder (planar waves)

Let $\Omega = \{0 \leq r \leq 1\} \times [-1, 1]$. *hp*-FEM+ADI: $O(np^3 \log p)$ solver.

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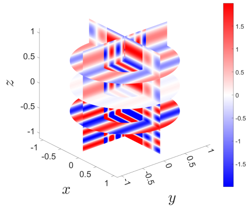

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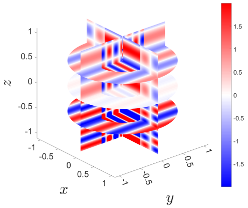

4 cells: *xy*-plane $\{0 \le r \le 1/2\}$ and $\{1/2 \le r \le 1\}$, *z*-plane $[-1, 0]$ and $[0, 1]$.

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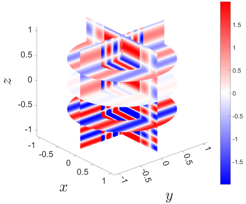

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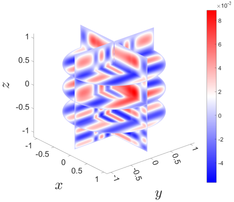
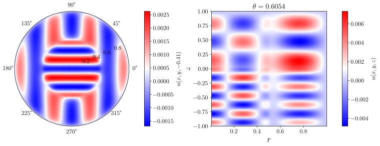
$u =$





(a) $z = -0.18$ (b) $\theta = 0.6$

Conclusions

- Introduced the first quasi-optimal complexity solver for the Helmholtz problem on the disk with radial discontinuities in the data.
- Based on a hierarchical *hp*-FEM consisting of Zernike (annular) polynomials.
- The stiffness and mass matrices are block diagonal where each block admits an optimal complexity factorization.
- A tensor-product basis constructs a basis for 3D cylinders.
- Utilizing ADI provides the first fast solver for the screened Poisson equation with discontinuous data in 3D cylinders.
- Can handle inhomogeneous Dirichlet/Neumann/mixed boundary conditions.

Extensions

fractional wave propagation, Schrödinger equation, eigenvalue problems.

Manuscript

I. P. A. Papadopoulos, S. Olver, *A sparse hierarchical hp-finite element method on disks and annuli*, `https://arxiv.org/abs/2402.12831` (2024).

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Thank you for listening!

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