Imperial College London

23 August 2023

Sparse spectral methods for fractional PDEs

ICIAM 2023: CT048

    

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[1]Imperial College London; [2]University of Oxford; [3]UCL

Are fractional PDEs physical?

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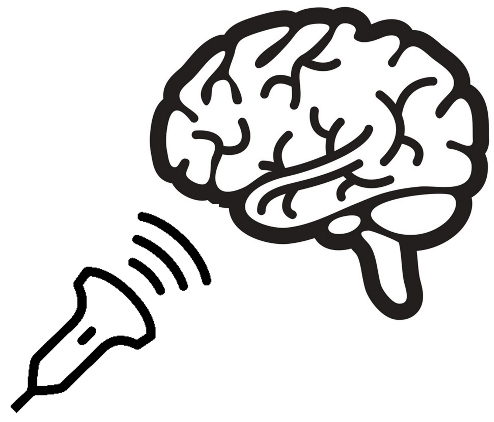

FPDEs describe wave absorption in the brain[1].

[1] Images from https://clipart.world/brain-clipart/black-and-white-brain-clipart/, https://www.kindpng.com/imgv/iRoiRR_sound-wave-clipart-ultrasound-ultrasound-clip-art-hd/.

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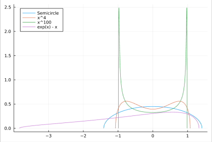
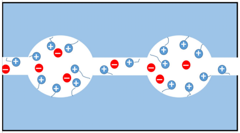
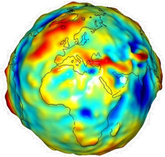
Other applications?



Equilibrium measures[2]

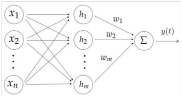


Dispersive transport of ions[3]



Long-range geophysical effects[5]

Image denoising[4]



Fully-layer connected neural networks[6]

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2 EquilibriumMeasures.jl

3 Alabi, Adetunji, et al. npj Clean Water 1.1 (2018): 10.

4 Ren, Zemin, Chuanjiang He, and Qifeng Zhang. Signal Processing 93.9 (2013): 2408-2421.

5 https://planetary-science.org/planetary-science-3/geophysics/

6 Zhang, Xuefeng, and Wenkai Huang. Fractal and Fractional 4.4 (2020): 50.

Fractional PDEs

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Observation

Solutions of fractional PDEs are “nonlocal” and may exhibit singularities.

Consequence

The solutions can be difficult to approximate numerically.

Challenge

How do we compute them with fast convergence?

Our proposal

A spectral method based on a so-called sum space.

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The problem

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The PDE

Find $u \in H^s(\mathbb{R})$, $s \in (0,1)$, that satisfies, for $\lambda \in \mathbb{R}$:

$$(\lambda \mathcal{I} + (-\Delta)^s)u = f. \quad \text{(fractional Helmholtz)}$$

$H^s(\mathbb{R})$

We seek solutions u that decay sufficiently quickly as $|x| \to \infty$. In particular

$$\|u\|_{H^s(\mathbb{R})} := \left(\int_{\mathbb{R}} u^2 \,\mathrm{d}x + \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}} \,\mathrm{d}x\mathrm{d}y \right)^{1/2} < \infty.$$

$\|\cdot\|_{H^s(\mathbb{R})}$ interpolates between $\|\cdot\|_{L^2(\mathbb{R})}$ and $\|\cdot\|_{H^1(\mathbb{R})}$.

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$(-\Delta)^s$

Ten (or more) equivalent definitions of the fractional Laplacian over $\mathbb{R}^d$. E.g. for $s \in (0,1)$,

$$(-\Delta)^s u(x) \coloneqq c_{d,s} \fint_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}} \, \mathrm{d}y$$

or

$$\mathcal{F}[(-\Delta)^s u](\omega) = |\omega|^{2s} \mathcal{F}[u](\omega).$$

We will focus on the special case $s = 1/2$.

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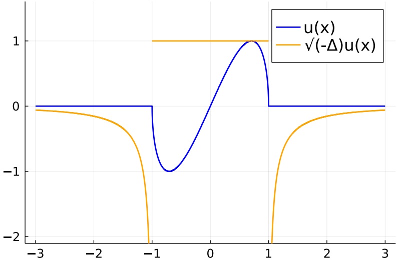

Nonlocal

$u(x) = 0$ for $|x| \geq 1$ but $(-\Delta)^{1/2}u(x) \neq 0$ for all $x \in \mathbb{R}$.

Singularities

As $x \downarrow 1$ and $x \uparrow -1$, then $|(-\Delta)^{1/2}u(x)| \to \infty$.

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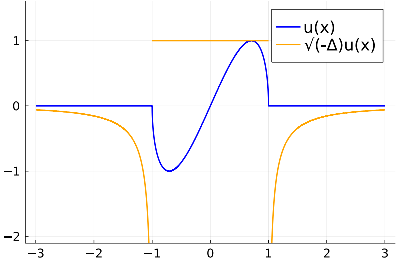

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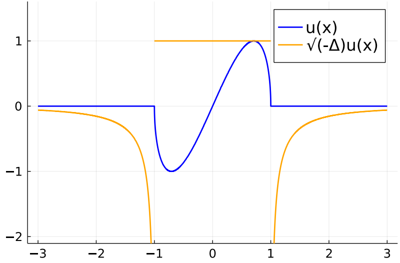

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Spectral methods

Consider the *ChebyshevT* polynomials, denoted $T_n(x)$. These satisfy

$$\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = \delta_{nm};\ T_0(x) = 1,\ \ T_1(x) = x,\ \ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

For $x \in [-1, 1]$, consider the approximation: $\mathrm{e}^{-x^2}\sin(x) \approx \sum_{k=0}^{n} f_k\,T_k(x)$.

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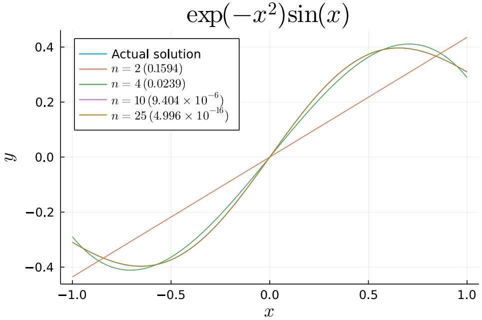

Dense spectral methods

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Many spectral methods for differential equations induce *dense* matrices ✗. Consider solving, on $[-1, 1]$,

$$-u'(x) = f(x), \quad u(-1) = 0.$$

A spectral method recipe

1. Expand $f(x)$ in the ChebyshevT polynomial basis, truncate, and collect the coefficients in vector $\mathbf{f}$.
2. Construct the derivative matrix D via a collocation method. D is **dense**.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the ChebyshevT expansion of $u(x)$.

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Sparse spectral methods

Let $\{U_n\}$ denote the ChebyshevU polynomials ortho. to $\sqrt{1-x^2}$.

An observation

For $n \geq 1$, $T_n'(x) = nU_{n-1}(x)$. Or in *quasimatrix* notation:

$$\begin{pmatrix} T_0'(x) & T_1'(x) & T_2'(x) & \dots \end{pmatrix} \begin{pmatrix} 0 & 1 & & \\ & & 2 & \\ & & & \ddots \end{pmatrix} = \begin{pmatrix} U_0(x) & U_1(x) & U_2(x) & \dots \end{pmatrix}$$

A sparse spectral method recipe (generalizeable to ODEs)

1. Expand $f(x)$ in the ChebyshevU polynomial basis, truncate, and collect the coefficients in vector $\mathbf{f}$.
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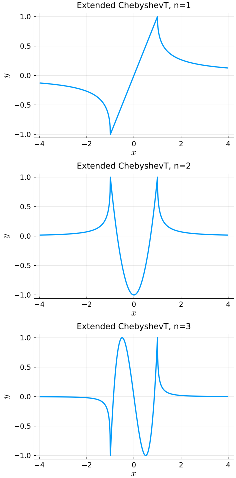
Extended Chebyshev functions

For $n \geq 1$,

$$\tilde{T}_n(x) := \begin{cases} T_n(x) & |x| \leq 1, \\ (x - \operatorname{sgn}(x)\sqrt{x^2 - 1})^n & |x| > 1. \end{cases}$$

$$\tilde{U}_n(x) := \begin{cases} U_n(x) & |x| \leq 1, \\ 2\tilde{T}_n(x) + \tilde{U}_{n-2}(x) & |x| > 1. \end{cases}$$

where $\tilde{U}_{-1}(x) := \begin{cases} 0 & |x| \leq 1, \\ -\frac{\operatorname{sgn}(x)}{\sqrt{x^2-1}} & |x| > 1, \end{cases}$



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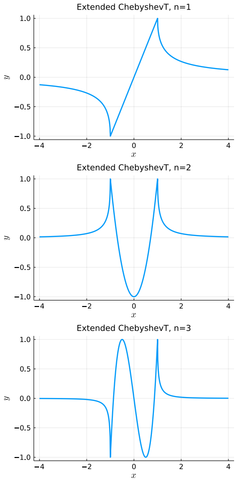

A sparse spectral method for an FPDE

$$W_n(x) \coloneqq (1-x^2)_+^{1/2} U_n(x), \; V_n(x) \coloneqq (1-x^2)_+^{-1/2} T_n(x).$$

$(-\Delta)^{1/2}$

$$(-\Delta)^{1/2} W_n(x) = (n+1)\tilde{U}_n(x),$$
$$(-\Delta)^{1/2} \tilde{T}_n(x) = nV_n(x).$$

Identity

$$W_n(x) = \frac{1}{2}[V_n(x) - V_{n+2}(x)],$$
$$\tilde{T}_n(x) = \frac{1}{2}[\tilde{U}_n(x) - \tilde{U}_{n-2}(x)].$$

Observation: The relationships are banded!

A sparse spectral method for an FPDE

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$$W_n(x) \coloneqq (1-x^2)_+^{1/2} U_n(x), \; V_n(x) \coloneqq (1-x^2)_+^{-1/2} T_n(x).$$

$(-\Delta)^{1/2}$

$$(-\Delta)^{1/2} W_n(x) = (n+1)\tilde{U}_n(x),$$
$$(-\Delta)^{1/2} \tilde{T}_n(x) = nV_n(x).$$

Identity

$$W_n(x) = \frac{1}{2}[V_n(x) - V_{n+2}(x)],$$
$$\tilde{T}_n(x) = \frac{1}{2}[\tilde{U}_n(x) - \tilde{U}_{n-2}(x)].$$

Observation: The relationships are banded!

A sparse spectral method for an FPDE

Key idea: use the sum space $\{\tilde{T}_n\} \cup \{W_n\}$.

$\lambda\mathcal{I} + (-\Delta)^{1/2}$

$$\underbrace{\{\tilde{T}_n\} \cup \{W_n\}}_{\text{sum space, } S} \xrightarrow{\lambda\mathcal{I}+(-\Delta)^{1/2}} \underbrace{\{\tilde{U}_n\} \cup \{V_n\}}_{\text{dual sum space, } S^*} .$$

A sparse spectral method recipe

1. Expand f in the dual sum space $f(x) \approx S^*(x)\mathbf{f}$.
2. Assemble the sparse matrix D induced by $(\lambda\mathcal{I} + (-\Delta)^{1/2})$.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$, $u(x) \approx S(x)\mathbf{u}$.

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A sparse spectral method recipe

1. Expand f in the dual sum space $f(x) \approx S^*(x)\mathbf{f}$.
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Expansion of the right-hand side f

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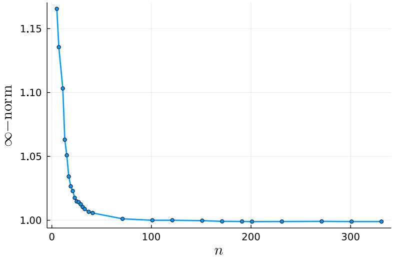
- Only expand f in $V_n^i(x)$ or W_n^i via the DCT.
- Solve a least squares collocation problem via a truncated SVD. [Backed by *frame* theory].



l^∞-norm of the coefficient vector for
$$(1-2x)\mathrm{e}^{-x^2} - \frac{\mathrm{i}}{x}\left(\mathrm{e}^{-x^2}|x|\,\mathrm{erf}(\mathrm{i}|x|)\right) + \frac{2}{\sqrt{\pi}}{}_1F_1(1;1/2;-x^2)$$

Example: the Gaussian

$$(\mathcal{I} + (-\Delta)^{1/2})u(x) = \mathrm{e}^{-x^2} + \frac{2}{\sqrt{\pi}}\,{}_1F_1(1; 1/2; -x^2).$$

${}_1F_1$ is the Kummer confluent hypergeometric function.

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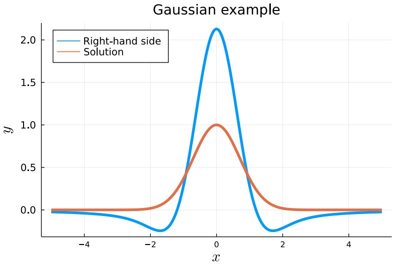

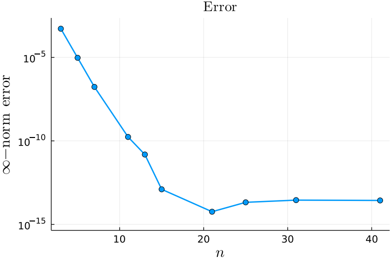

Example: wave propagation

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Consider the FPDE ($u(x,t) \to 0$ as $|x| \to \infty$):

$$[(-\Delta)^{1/2} + \mathcal{H} + \frac{\partial^2}{\partial t^2}]u(x,t) = (1-x^2)_+^{1/2} U_4(x) \mathrm{e}^{-t^2}.$$

A Fourier transform in time gives ($\hat{u}(x,\omega) \to 0$ as $|x| \to \infty$):

$$[(-\Delta)^{1/2} + \mathcal{H} - \omega^2]\hat{u}(x,\omega) = \sqrt{\pi}(1-x^2)_+^{1/2} U_4(x) \mathrm{e}^{-\omega^2/4}.$$

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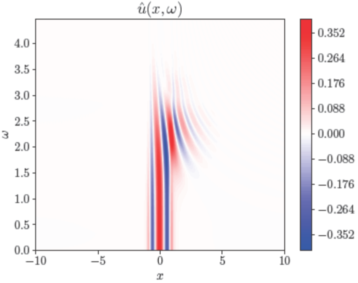

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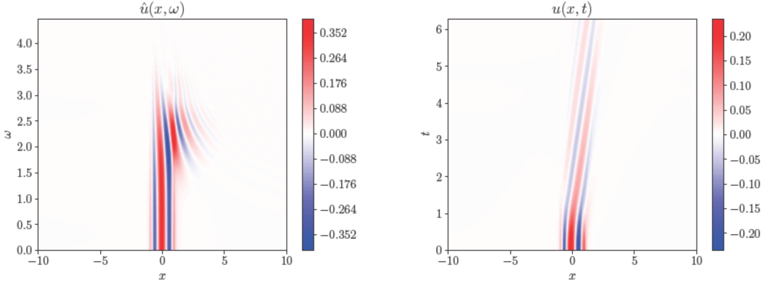

Conclusions

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- A sparse spectral method for solving the identity + sqrt-Laplacian;
- Based on a carefully chosen sum space;
- Implementation written in `Julia` see `https://github.com/ioannisPApapadopoulos/SumSpaces.jl`.

A sparse spectral method for fractional differential equations in one-spacial dimension

I.P., S. Olver, 2022, arXiv preprint arXiv:2210.08247

Ongoing work

- Generalization to $(-\Delta)^s$ with $s \in (0,1)$ utilizing extended/weighted Jacobi polynomials;
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Thank you for listening!

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