A frame approach for equations involving the fractional Laplacian

John Papadopoulos[1], Timon Gutleb[2], José Carrillo[3], Sheehan Olver[4]
[1]Weierstrass Institute Berlin, [2]British Columbia,[3]Oxford, [4]Imperial

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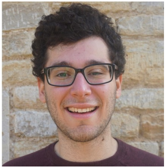


The fractional Laplacian

Denoted by $(-\Delta)^s$

Ten (or more) equivalent definitions of the fractional Laplacian over $\mathbb{R}^d$. E.g. for $s \in (0,1)$,

As a singular integral

$$(-\Delta)^s u(x) := c_{d,s} \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}} \mathrm{d}y$$

As a Fourier multiplier

$$\mathcal{F}[(-\Delta)^s u](\omega) = |\omega|^{2s} \mathcal{F}[u](\omega).$$

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Equations of interest

$$\mathcal{L}u = f, \quad u(x) \to 0 \text{ as } |x| \to \infty.$$

Fractional screened Poisson

$$(\mathcal{I} + (-\Delta)^s)u = f.$$

Fractional screened Poisson with multiple exponents

$$(\mathcal{I} + (-\Delta)^{s_1} + (-\Delta)^{s_2})u = f.$$

Fractional heat

$$(\partial_t + (-\Delta)^s)u = f.$$

Fractional heat with variable exponent

$$(\partial_t + (-\Delta)^{s(t)})u = f.$$

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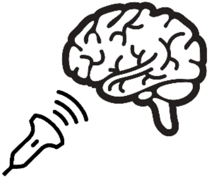
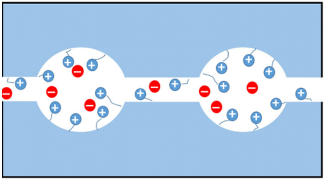
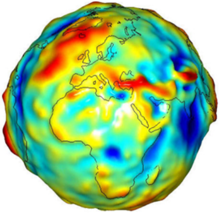


Wave absorption



Dispersive transport of ions

Image denoising



Long-range geophysical effects

Core idea

$$\mathcal{L}u = f.$$

$\Phi_N = \{\phi_i\}_{i=1}^N$ fin. dim. basis for the solution u.

Apply operator $\psi_i := \mathcal{L}\phi_i, \; i = 1, \ldots, N$.

$\Psi_N := \{\psi_i\}_{i=1}^N$ fin. dim. basis for the right-hand side f.

$$u(x) = \mathcal{L}^{-1} f(x) \approx \mathcal{L}^{-1} \sum_{i=1}^N c_i \psi_i(x) = \sum_{i=1}^N c_i (\mathcal{L}^{-1}\psi_i)(x) = \sum_{i=1}^N c_i \phi_i(x).$$

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Frame solver

Algorithm for frame solver

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   - $\mathcal{L}, f$. $\triangleright$ Fractional PDE operator and right-hand side.
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   - ϵ, x_j, $j \in \{1 : M\}$. $\triangleright$ SVD tolerance and collocation points.
2. Output:
   - $\mathbf{u} \in \mathbb{R}^N$. $\triangleright$ $u(x) \approx \sum_{i=1}^N \mathbf{u}_i \phi_i(x)$.
3. Assemble the matrix $X \in \mathbb{R}^{N \times M}$, $X_{ij} = (\mathcal{L}\phi_i)(x_j)$, $i = 1 : N$, $j = 1 : M$.
4. Assemble the vector $\mathbf{y}_j = f(x_j)$, $j = 1 : M$.
5. Via an ϵ-truncated SVD projection, compute $\mathbf{f} \approx \underset{\mathbf{v} \in \mathbb{R}^N}{\operatorname{argmin}} \|X\mathbf{v} - \mathbf{y}\|_{\ell^2}$.
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Does an ϵ-truncated SVD projection result in a well-behaved expansion?

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Frames

A frame is a family of functions with redundancy. More explicitly:

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Consider a Hilbert space $(H, (\cdot,\cdot)_H)$. $\Psi = \{\psi_n\}$, $\psi_n \in H$ is a *frame* for H if there exist constants $0 < c \leq C < \infty$ s.t.

$$c\|g\|_H^2 \leq \sum_n |(g, \psi_n)_H|^2 \leq C\|g\|_H^2 \text{ for all } g \in H.$$

Frame expansions

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Constructing a solver with frames

Can we efficiently expand f in $\Psi := \mathcal{L}\Phi$?

Theorem

Suppose that $\Phi = \{\phi_n\}$ is a frame for H. If the adjoint operator $\mathcal{L}^*$ is bounded and positive-definite in H then $\Psi = \mathcal{L}\Phi = \{\mathcal{L}\phi_n\}$ is a frame for the dual space H^*.

Solver pipeline

1. Pick a frame Φ for H in which we will expand the solution $u(x)$.
2. Compute the dual frame Ψ, i.e. compute $\mathcal{L}\phi_n$ for each $n \in \mathbb{N}$.
3. Expand $f(x)$ in the frame Ψ, $f(x) = \sum_i \mathbf{f}_i \psi_i(x)$.
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Suppose that $\Phi = \{\phi_n\}$ is a frame for H. If the adjoint operator $\mathcal{L}^*$ is bounded and positive-definite in H then $\Psi = \mathcal{L}\Phi = \{\mathcal{L}\phi_n\}$ is a frame for the dual space H^*.

Solver pipeline

1. Pick a frame Φ for H in which we will expand the solution $u(x)$.
2. Compute the dual frame Ψ, i.e. compute $\mathcal{L}\phi_n$ for each $n \in \mathbb{N}$.
3. Expand $f(x)$ in the frame Ψ, $f(x) = \sum_i \mathbf{f}_i \psi_i(x)$.
4. Then $u(x) = \sum_i \mathbf{f}_i \phi_i(x)$.

What do we pick as Φ in the context of the fractional Laplacian?

Constructing a solver with frames

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Exceptionally elegant formulae

Weighted Jacobi polynomials

$$Q_n^{(a,b)}(x) := (1-x)^a(1+x)^b P_n^{(a,b)}(x) \text{ for } x \in [-1,1].$$

Define $Q_n^{(a,b)}(x) = 0$ for all $x \in \mathbb{R}\backslash[-1,1]$.

The ${}_2F_1$ hypergeometric function

$${}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} \text{ for } x \in (-1,1),$$

where $(q)_n := q(q+1)\cdots(q+n-1)$ for $n > 1$ and $(q)_0 = 1$.

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$(-\Delta)^s Q_n^{(a,a)}$, $s \in (-1/2, 0) \cup (0, 1)$

$$(-\Delta)^{s} Q_n^{(a,a)}(x) = \begin{cases} \text{constant } \times x^c \times {}_2F_1 \text{ function} & |x| < 1, \\ \text{constant } \times |x|^d \times {}_2F_1 \text{ function} & |x| > 1. \end{cases}$$

More specifically:

$$(-\Delta)^{s} Q_n^{(a,a)}(x) = 4^s \frac{\Gamma(a+n+1)}{n!} x^{n-2\lfloor \frac{n}{2} \rfloor}$$

$$\times \begin{cases} \dfrac{\pi \, {}_2F_1\left(-a+s-\lfloor \frac{n}{2} \rfloor, n+s-\lfloor \frac{n}{2} \rfloor+\frac{1}{2}; n-2\lfloor \frac{n}{2} \rfloor+\frac{1}{2}; x^2\right)}{\sin\left(\pi\left(2\lfloor \frac{n}{2} \rfloor-n-s+\frac{1}{2}\right)\right)\Gamma\left(n-2\lfloor \frac{n}{2} \rfloor+\frac{1}{2}\right)\Gamma\left(-n-s+\lfloor \frac{n}{2} \rfloor+\frac{1}{2}\right)\Gamma\left(a-s+\lfloor \frac{n}{2} \rfloor+1\right)} & |x| < 1, \\ -\dfrac{2^{-n-2s}\sin(\pi s)\Gamma(n+2s+1)|x|^{-2\lfloor \frac{n-1}{2} \rfloor-2s-3} \, {}_2F_1\left(s+\lfloor \frac{n}{2} \rfloor+1, \frac{2\lfloor \frac{n-1}{2} \rfloor+3}{2}+s; \frac{2n+3}{2}+a; \frac{1}{x^2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2n+3}{2}+a\right)} & |x| > 1. \end{cases}$$

Leibniz Association

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For $d > 1$ equivalent formulae exist for weighted Zernike polynomials.

Explicit fractional Laplacians and Riesz potentials of classical functions

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Solution frame

Consider $\mathcal{L} = \mathcal{I} + (-\Delta)^s$.

Φ

$$\Phi = \{Q_n^{(s,s)}, (-\Delta)^{-s} Q_n^{(-s,-s)}\}_{n\in\mathbb{N}}.$$

Theorem

Φ is a frame on $H_w^s(\mathbb{R}) = \{u \in L_w^2(\mathbb{R}) : \mathrm{supp}(u) \subseteq \mathrm{supp}(w)\ (-\Delta)^{s/2} u \in L^2(\mathbb{R})\}$ with $w(x) = (1-x^2)_+^{-s}$.

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An important ingredient to a successful solver is including several translations of Φ in the overall solution frame.

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Multiple exponents

$$(\mathcal{I} + (-\Delta)^{1/3} + (-\Delta)^{1/5})u(x) = \mathrm{e}^{-x^2} + (-\Delta)^{1/3}\mathrm{e}^{-x^2} + (-\Delta)^{1/5}\mathrm{e}^{-x^2}.$$

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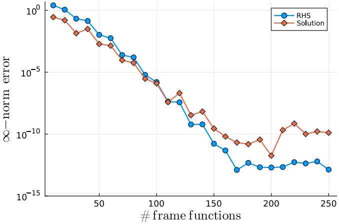

(a) ℓ^∞-norm error.

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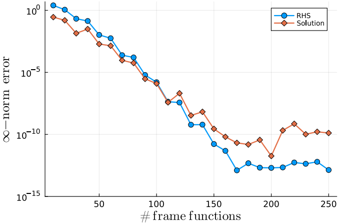

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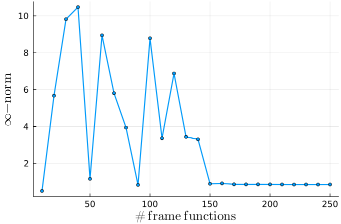

(b) ℓ^∞-norm of the coefficient vector.

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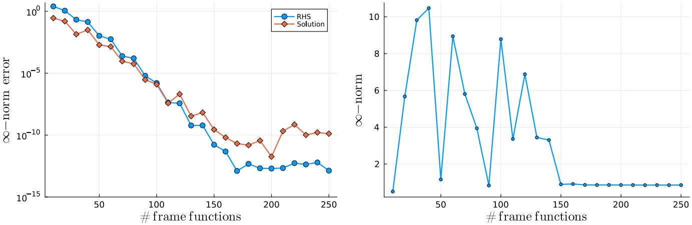


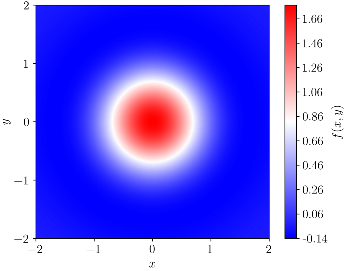
(a) ℓ^∞-norm error. (b) ℓ^∞-norm of the coefficient vector.

$$\Phi = \bigcup_{k=1}^{5} \{(-\Delta)^{-1/4} Q_j^{I_k,(-1/4,-1/4)}\}_{j=0}^{\infty} \cup \{Q_j^{I_k,(1/4,1/4)}\}_{j=0}^{\infty}$$

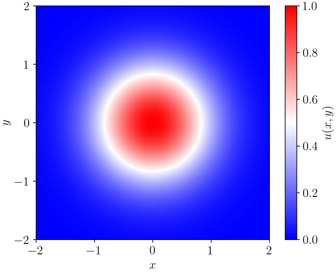
where $I_1, \ldots, I_5$ are $[-5,-3]$, $[-3,-1]$, $[-1,1]$, $[1,3]$, and $[3,5]$, respectively.

Two-dimensional Gaussian

$$(-\Delta)^{1/2}u(x,y) = 2\Gamma(3/2)\,{}_1F_1(3/2;1;-x^2-y^2).$$



(a) $f(x,y)$



(b) $u(x,y)$

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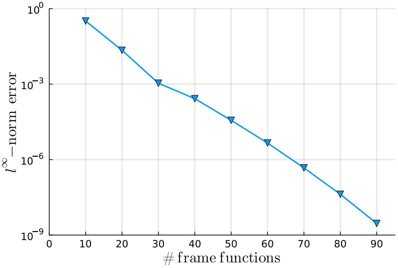

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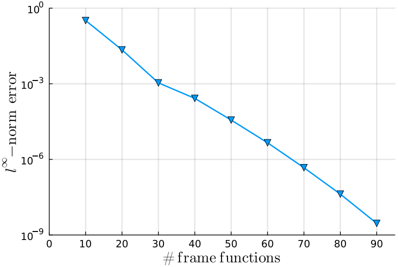

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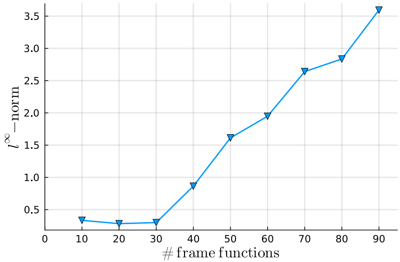

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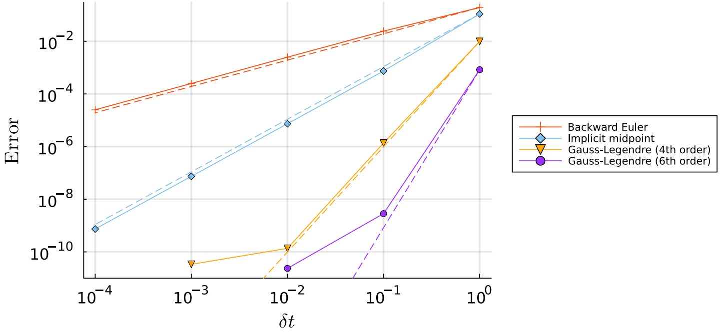
Fractional heat

$$(\partial_t + (-\Delta)^{1/2})u(x,t) = 0, \quad u(x,0) = (1+x^2)^{-1}.$$



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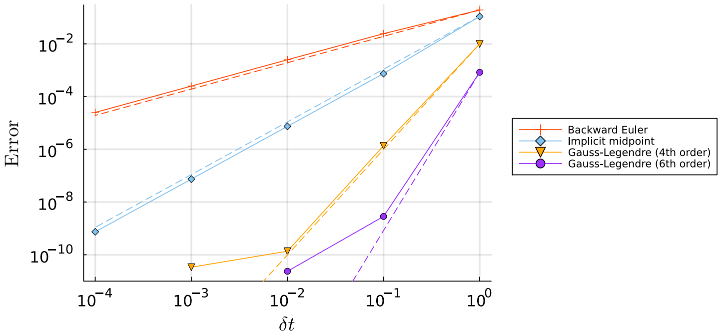

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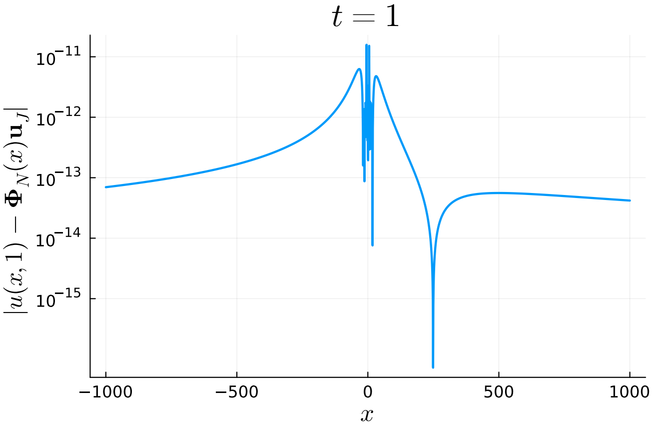
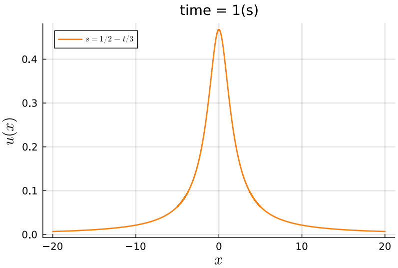
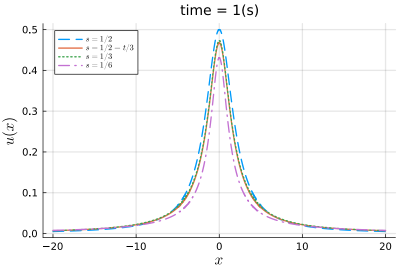


Fractional heat with variable exponent

$$(\partial_t + (-\Delta)^{s(t)})u(x,t) = 0, \quad s(t) = 1/2 - t/3, \quad u(x,0) = (1+x^2)^{-1}.$$





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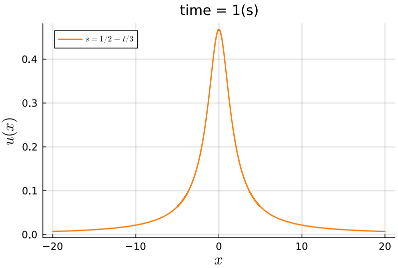

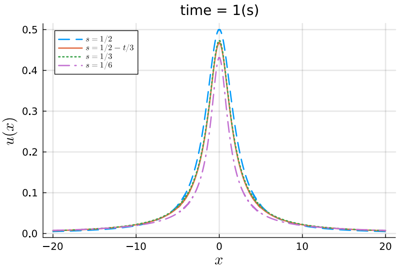

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- Turn the problem from a solve to an expansion in a non-standard finite-dimensional basis.
- Use an ϵ-truncated SVD projection to obtain expansion coefficients.

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arXiv, 2023. `https://arxiv.org/abs/2311.12451`.

FractionalFrames.jl

`https://github.com/ioannisPApapadopoulos/FractionalFrames.jl`.

Leibniz Association

Unfinished business

Faster assembly of X?

Implementation of three-term recurrence for $|x| < 1$ and (F. W. J) Olver's and Miller's algorithm for evaluation in $|x| > 1$.

Faster ϵ-truncated SVD projection in 2D?

Perhaps via the AZ-algorithm?

Frame in a more standard Hilbert space?

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Thank you for listening!

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