A semismooth Newton method for obstacle-type quasivariational inequalities

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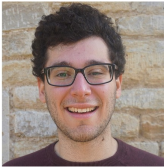


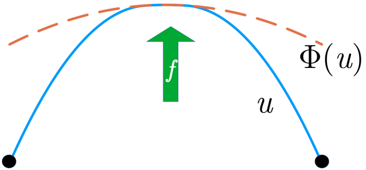

What are obstacle-type QVIs?

Consider the constraint set: $K(u) := \{v \in H^1_0(\Omega) : v \leq \Phi(u) \text{ a.e.}\}$

$$\min_{u \in K(u)} \int_\Omega \frac{1}{2}|\nabla u|^2 - f \cdot u \,\mathrm{d}x.$$

First-order optimality condition is an obstacle-type QVI

Find $u \in H^1_0(\Omega) : (\nabla u, \nabla(v-u))_{L^2(\Omega)} \geq (f, v-u)_{L^2(\Omega)}$ for all $v \in K(u)$.



Sandpiles, thermoforming, elastic bilayers, image processing, option pricing, fluid flow with pressure constraints... Why doesn't everyone model with QVIs? ...they are very hard to solve!

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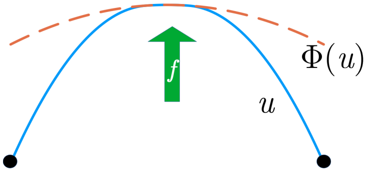

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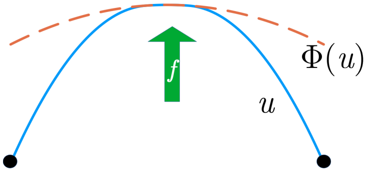

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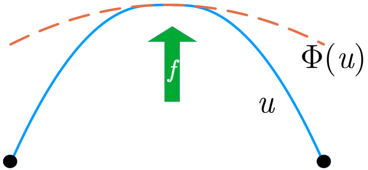

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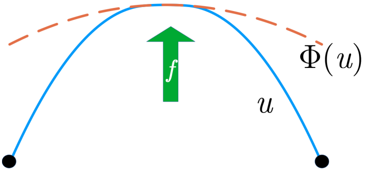

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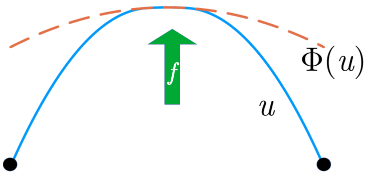

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Semismooth Newton method

Rewrite the QVI as the fixed point problem $u = S(\Phi(u))$.

Obstacle map $S : H_0^1(\Omega) \to H_0^1(\Omega)$, $S : \phi \mapsto u_\phi$

$u_\phi = S(\phi)$ maps from the obstacle $\to$ solution of the obstacle VI, i.e.

$$(\nabla u_\phi, \nabla(v - u_\phi))_{L^2(\Omega)} \geq (f, v - u_\phi)_{L^2(\Omega)} \text{ for all } v \in H_0^1(\Omega),\ v \leq \phi.$$

⚠Evaluating $\Phi(u)$ might require a nonlinear PDE solve.

⚠Evaluating $S(\phi)$ requires a VI solve.

SSN step for the QVI

Let $R(u) = u - S(\Phi(u))$. The SSN iteration is $u_{i+1} = u_i + \delta$ where

$$G_R(u_i)\delta = -R(u_i).$$

Leibniz Association

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Chain rule

$$G_R(u_i) = \mathrm{Id} - G_S(\Phi(u_i))G_\Phi(u_i).$$

Theorem

The obstacle map $S : \phi \mapsto u_\phi$ is *semismooth* with a Newton derivative G_S where $\|G_S\| \leq 1$. Moreover,

$$G_S(\phi)\zeta = \zeta + z_\zeta$$

where $z_\zeta \in H^1_0(\mathcal{I}(\phi))$ satisfies

$$(\nabla z_\zeta - \nabla\zeta, \nabla v) = 0 \;\text{ for all } v \in H^1_0(\mathcal{I}(\phi))$$

and $\mathcal{I}(\phi) = \{x \in \Omega : S(\phi)(x) < \phi(x) \text{ a.e.}\} \subseteq \Omega.$

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Deriving the SSN System

SSN System

SSN update δ satisfies

$$[\mathrm{Id} - G_S(\Phi(u_i))G_\Phi(u_i)]\delta = -R(u_i).$$

Introduce auxiliary variables $\eta = G_\Phi(u_i)\delta$ and $\mu = G_S(\Phi(u_i))\eta - \eta$.
Reformulate SSN system as

$$\begin{pmatrix} \mathrm{Id} & -\mathrm{Id} & -\mathrm{Id}|_{\mathcal{I}(u_i)} \\ G_\Phi(u_i) & -\mathrm{Id} & 0 \\ 0 & (G_S(\Phi(u_i)) - \mathrm{Id})|_{\mathcal{I}(u_i)} & -\mathrm{Id}|_{\mathcal{I}(u_i)} \end{pmatrix} \begin{pmatrix} \delta \\ \eta \\ \mu \end{pmatrix} = -\begin{pmatrix} R(u_i) \\ 0 \\ 0 \end{pmatrix}.$$

where $\mathcal{I}(u_i) := \{x \in \Omega : S(\Phi(u_i))(x) < \Phi(u_i)(x) \text{ a.e.}\}$.

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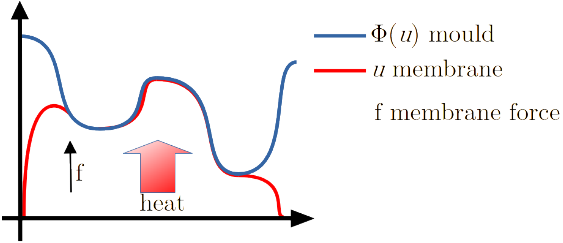
Thermoforming: an obstacle-type QVI



Model

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with T as the solution of

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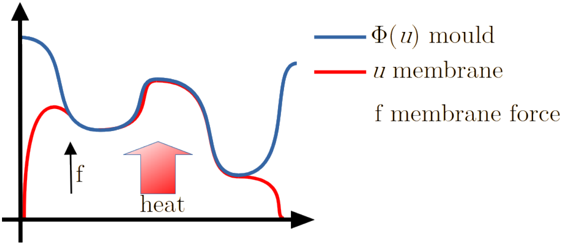

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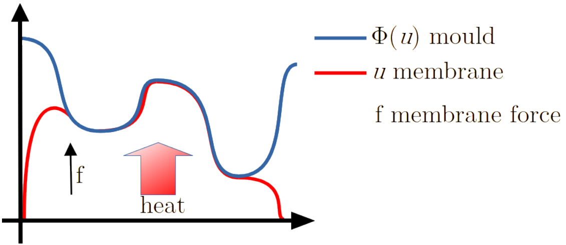

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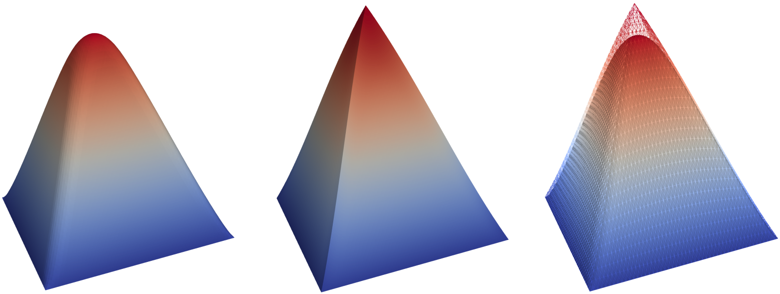
Example: setup

$$\Omega = (0,1)^2, \qquad \Phi_0(x_1,x_2) = 1 - 2\max(|x_1 - 0.5|, |x_2 - 0.5|),$$

$$f(x_1,x_2) = 25, \qquad \psi(x_1,x_2) = \sin(\pi x_1)\sin(\pi x_2), \qquad k = 1,$$

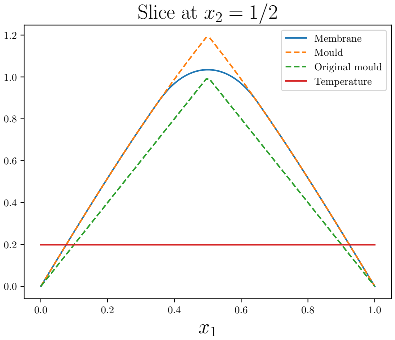
$$g(s) = \begin{cases} 1/5 & \text{if } s \leq 0, \\ (1-s)/5 & \text{if } 0 < s < 1, \\ 0 & \text{otherwise.} \end{cases}$$



(a) Membrane u (b) Mould $\Phi_0 + \psi T$ (c) Membrane & Mould

Example: convergence

- CG_1 FEM discretization for u and T.
- Active set in SSN step implemented via one step of `vinewtonrsls`.
- VI solver for evaluating $S(\phi)$: path-following Moreau-Yosida regularization (PFMY) + feasibility restoration with `vinewtonrsls`.
- Fixed point method: $u_{i+1} = S(\Phi(u_i))$ [Converges linearly].



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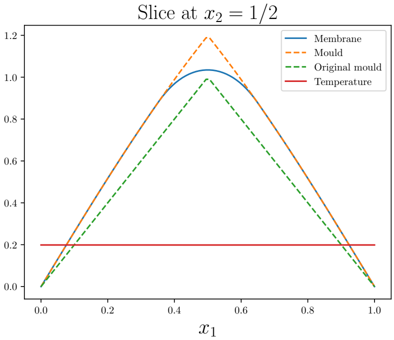

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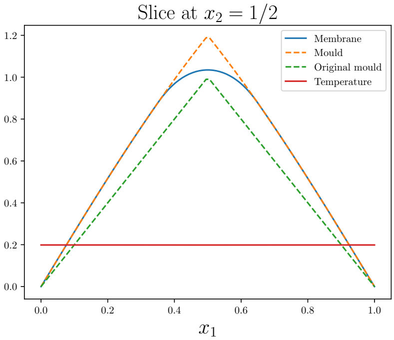

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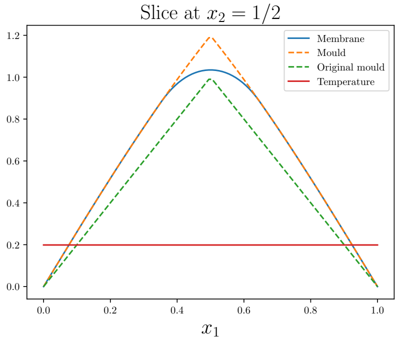

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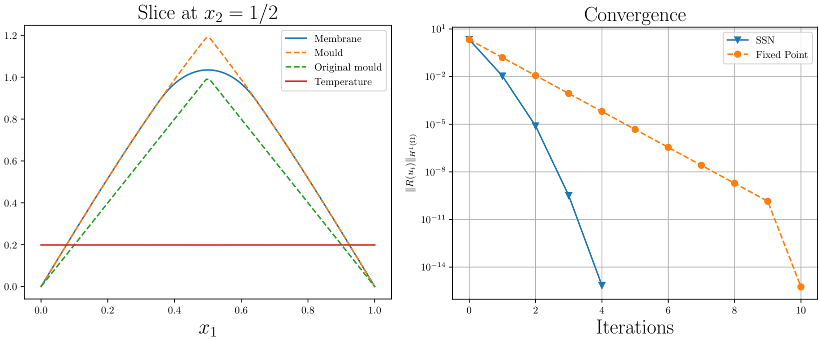

Example: mesh independence

	Outer loop	Evaluate Φ	Evaluate S	
h	SSN	Newton	PFMY	$+$`vinewtonrsls`
0.04	4	8	84	11
0.02	4	8	79	10
0.01	4	8	79	15
0.00667	4	8	79	19
0.005	4	9	79	20
0.004	4	8	79	16
0.00333	3	9	64	17

Table: Mesh independence of the SSN.

Conclusions

- A semismooth Newton method for solving obstacle-type QVIs;
- An active-set strategy implemented in Firedrake;
- Theory relies on recent semismooth results for the obstacle map S.

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A. Alphonse, C. Christof, M. Hintermüller, I. P. A. Papadopoulos, to appear, 2024.

semismoothQVIs

`https://github.com/ioannisPApapadopoulos/semismoothQVIs`.

Conclusions

- A semismooth Newton method for solving obstacle-type QVIs;
- An active-set strategy implemented in Firedrake;
- Theory relies on recent semismooth results for the obstacle map S.

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`Depending on your platform, PETSc may take an hour or more to build!`

Thank you for listening!

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